


L. Cameron

COMMERCIAL ARITHMETIC

FREDERICK G.W. BROWN



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PREFACE

IN recent years, there has been a strong tendency to reduce the scope and simplify the substance of the teaching of arithmetic in schools. Both syllabuses and text-books now devote far less attention than formerly to aspects of the subject which involve intricate computations, especially those of a financial or commercial kind rarely likely to be required by general students when they enter life in the workshop or office. On this account, the course of work in most elementary text-books of arithmetic is now insufficient for those who wish to qualify in the subject for entry into many of the commercial professions. Whilst it is quite true that an intelligent study of the fundamental principles of arithmetic should develop the ability to apply those principles to the solution of the many practical problems arising from the progressive state of human activity, yet the student's equipment to-day must embrace much more than what was known formerly as pure arithmetic. This is especially true in the commercial world where, for instance, problems on interest, depreciation, the instalment plan, insurance, etc., have frequently to be dealt with. Not only are the fundamentals of arithmetic needed, but also it is necessary for the student to have a working and applied knowledge of algebra, notably of the progressions. A modern text-book of Commercial Arithmetic must therefore have an enlarged scope in order that the essential algebraic framework may be adequately treated and applied.

The present volume is an attempt to meet these requirements; and the course is designed to cover the syllabuses in the subject prescribed by the chief examining authorities in Commercial Arithmetic. The text is divided into two parts corresponding roughly to the usual courses specified for the Elementary and Higher (Intermediate and Advanced) Stages respectively. The treatment is related, to a

certain extent, especially in Part II, to the appropriate parts of the theory of commerce and, as a necessary sequence, most of the problems considered are of a commercial character. Algebraic methods have been freely used and the progressions dealt with in a practical form. The use of logarithms has been fully discussed and, as is essential in many monetary problems, seven-figure logarithms have been introduced. A convenient table of these is provided at the end.

Particular attention is given throughout to shortened methods of working and to their accuracy, and emphasis is laid on the importance of not carrying calculations beyond the figure where they cease to have a practical value.

A chapter on the mechanical aids to computation has been inserted at the end. This gives a brief sketch indicating how modern calculating machines are used in business.

Ample provision for the student's practice has been made by the graded exercises attached to each chapter, and by the typical examination papers given at the end. Many of the exercises are taken from recent examination papers; for permission to reproduce these, I am indebted to the following authorities—the Birmingham and Midland Institute, the Chartered Institute of Secretaries, the College of Preceptors, the London Chamber of Commerce, the Royal Society of Arts and the Union of Lancashire and Cheshire Institutes.

To Sir Richard Gregory, Bt., and to Mr. A. J. V. Gale, M.A., I owe a heavy debt of gratitude for their constant interest, help and expert advice at every stage in the production of the book.

I also wish to record my thanks to Mr. R. Holmes, M.Sc., not only for reading the proof sheets and making many valuable suggestions thereon, but also for undertaking the arduous task of checking the answers to the exercises; to Mr. F. W. Dent for considerable help in the preparation of the typescript; to the Burroughs Adding Machine Ltd. and Messrs. Felt and Tarrant for the use of the illustrations shown on pages 313–315; to the publishers for much assist-

ance and for the use of the tables of four-figure logarithms, which are taken from Castle's *Logarithmic and Other Tables for Schools*; and finally, to the printers for the excellence of their work.

It is more than likely that a few errors have even now escaped detection, and notification of these will be welcomed.

F. G. W. BROWN

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THE following abbreviations indicate the source of questions taken from recent Examination Papers :

B.M.I. = Birmingham and Midland Institute.

C.I.S. = Chartered Institute of Secretaries.

C.P. = College of Preceptors.

L.Ch.C. = London Chamber of Commerce.

R.S.A. = Royal Society of Arts.

U.L.C.I. = Union of Lancashire and Cheshire Institutes.

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PART I

CHAPTER I

ORDINARY OR VULGAR FRACTIONS

1.1. Fractions.

WHEN a penny is divided into four equal parts, each part is known as a farthing, and is denoted, as everyone knows, by $\frac{1}{4}$ d. This is a convenient method of denoting the division ($1 \div 4$) pence. Similarly, when 3d. is divided into four equal parts, i.e. ($3 \div 4$) pence, the result is written $\frac{3}{4}$ d. Again, 5 lb. divided by 8 becomes $\frac{5}{8}$ lb., which implies ($5 \div 8$) lb.

The quantities $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{8}$ are called **fractions**, because they each denote a part of the unit concerned. Since

$$\frac{1}{4} = 1 \div 4; \quad \frac{3}{4} = 3 \div 4; \quad \frac{5}{8} = 5 \div 8,$$

a fraction really represents *division*.

Ex. 1. *In exchanging 16 Valparaiso pesos into English money, a traveller received 19s. 1d.; find the value of a peso in pence.*

Since 16 pesos are equivalent to 19s. 1d., i.e. 229 pence, 1 peso will be equivalent to ($229 \div 16$) pence,

i.e. to $\frac{229}{16}$ pence.

On dividing 229 by 16, the quotient is 14 with a remainder of 5, and 5 divided by 16 is written $\frac{5}{16}$.

Hence, the value of 1 peso is $14\frac{5}{16}$ pence.

Note that $\frac{5}{16} = \frac{4+1}{16} = \frac{4}{16} + \frac{1}{16}$, and 4 divided by 16 is the same as 1 divided by 4, so that

$$\frac{5}{16}\text{d.} = \frac{1}{4}\text{d.} + \frac{1}{16}\text{d.},$$

i.e. $\frac{5}{16}$ d. is larger than $\frac{1}{4}$ d., or a farthing, by $\frac{1}{16}$ d.

1·2. Ordinary Fractions.

Let AB (Fig. 1) represent a foot-rule showing the division into inches, then the length of any part AK can be expressed as a fraction

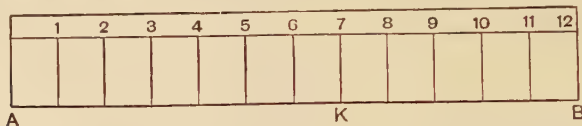


FIG. 1.

of the whole length of the rule, i.e. of 12 inches. Thus suppose $AK = 7$ inches, then the length of AK is exactly *seven-twelfths* of the length of AB . This fraction is written $\frac{7}{12}$ and indicates that the whole length of the rule is divided into 12 equal parts and that 7 of these parts are contained in the length of AK . Such a fraction is called an **Ordinary or Vulgar Fraction**, and consists of a **denominator** (Latin, *nomen* = a name) indicating the number of equal parts into which the whole is divided, and a **numerator** (Latin, *numerus* = a number) denoting the number of those parts taken. Hence, every ordinary fraction is of the form :

$$\frac{\text{numerator}}{\text{denominator}}.$$

Note that if AK had been 8 inches long, its length would be $\frac{8}{12}$ of a foot. But $\frac{8}{12} = \frac{2}{3}$, since 8 parts in 12 is exactly the same as 2 in 3. Similarly, 6 inches = $\frac{6}{12}$ ft. = $\frac{1}{2}$ ft.

Ex. 2. Find, to the nearest penny, the values of (i) $\pounds \frac{11}{14}$, (ii) $\frac{9}{16}$ of £3 8s. 9d.

(i) Since £1 = 240 pence,

$$\pounds \frac{1}{14} = 240 \div 14 \text{ pence, i.e. } \frac{240}{14} \text{ pence,}$$

$$\text{and } \pounds \frac{11}{14} = \pounds \frac{1}{14} \times 11 = \frac{240 \times 11}{14} \text{ pence}$$

$$= \frac{2640}{14} = 188\frac{8}{14} \text{ pence} = 15\text{s. } 8\frac{8}{14}\text{d.}$$

Now $\frac{8}{14} = \frac{7}{14} + \frac{1}{14} = \frac{1}{2} + \frac{1}{14}$, so that $\frac{8}{14}$ d. is greater than $\frac{1}{2}$ d. by $\frac{1}{14}$ d. Hence, $8\frac{8}{14}$ d. is nearer 9d. than 8d.

∴ to the nearest penny, $\pounds 1\frac{1}{4} = 15$ s. 9d.

(ii) $\frac{9}{16}$ of £3 8s. 9d. = $\frac{9}{16}$ of 825 pence

$$= \frac{825 \times 9}{16} \text{ pence} = \frac{7425}{16} \text{ pence}$$

$$= 464\frac{1}{16} \text{ pence} = \pounds 1 \text{ 18s. } 8\frac{1}{8}\text{d.}$$

But $8\frac{1}{8}$ d. is much nearer to 8d. than to 9d., so that the required value, to the nearest penny, is **£1 18s. 8d.**

1·3. Practical Approximation.

In commercial transactions, sums of money are usually calculated to the nearest penny, and in general, results of other concrete calculations are expressed to the nearest unit concerned; any fraction of the unit less than $\frac{1}{2}$ is rejected, whilst one equal to or greater than $\frac{1}{2}$ is taken as a complete unit. Thus $5\frac{1}{4}$ lb. is regarded as 5 lb., since $\frac{1}{4}$ is less than $\frac{1}{2}$, but $5\frac{3}{4}$ lb. is taken as 6 lb., since $\frac{3}{4}$ is greater than $\frac{1}{2}$. (See pp. 22-25.)

Ex. 3. *Express 12 cwt. 3 qr. 12 lb. as a fraction of a ton; hence, find the cost, to the nearest penny, of 12 cwt. 3 qr. 12 lb. of a commodity at £1 2s. 11d. per ton.*

1 ton = 2240 lb., and

$$\begin{aligned} 12 \text{ cwt. } 3 \text{ qr. } 12 \text{ lb.} &= (112 \times 12) + (28 \times 3) + 12 \text{ lb.} \\ &= 1344 + 84 + 12 \text{ lb.} = 1440 \text{ lb.} \end{aligned}$$

$$\therefore \text{ the required fraction} = \frac{1440}{2240} \text{ of a ton.}$$

$$\text{Now} \quad 1440 = 10 \times 144 = 10 \times 16 \times 9,$$

$$\text{and} \quad 2240 = 10 \times 224 = 10 \times 16 \times 14;$$

hence, if the product of the common factors 10×16 , or 160 be taken as one larger unit, $1440 = 9$ of these units and $2240 = 14$ of them. and the above fraction becomes $\frac{9}{14}$ of a ton.

From this result, it is evident that the cost of 12 cwt. 3 qr. 12 lb. at £1 2s. 11d. per ton

$$\begin{aligned}
 &= \frac{9}{14} \text{ of } £1 \text{ 2s. 11d.} = \frac{9}{14} \text{ of } 275 \text{ pence} \\
 &= \frac{9 \times 275}{14} \text{ pence} = \frac{2475}{14} \text{ pence} = 176\frac{11}{14} \text{ pence} \\
 &= 14\text{s. } 8\frac{1}{4}\text{d.} \\
 \therefore \text{ cost to the nearest penny} &= 14\text{s. 9d.}
 \end{aligned}$$

1.4. Cancelling. The Fundamental Rule.

From the first part of Ex. 3, the fraction $\frac{1440}{2240}$ might have been simplified directly by dividing both numerator and denominator by all factors common to them. This process is known as *reducing the fraction to its lowest terms*. The common factors thus removed are said to be cancelled.

Indeed, the fundamental rule in working with ordinary fractions may be stated as follows :

The value of a fraction is unaltered when both numerator and denominator are divided or multiplied by the same number.

$$\text{Thus,} \quad \frac{12}{15} = \frac{12 \div 3}{15 \div 3} = \frac{4}{5},$$

$$\text{and} \quad \frac{12}{15} = \frac{12 \times 8}{15 \times 8} = \frac{96}{120};$$

$$\text{hence} \quad \frac{96}{120}, \quad \frac{12}{15} \text{ and } \frac{4}{5} \text{ are all equal.}$$

To verify this, note that

$$£\frac{96}{120} = £\frac{1}{120} \times 96 = 2\text{d.} \times 96 = 16\text{s.}$$

$$£\frac{12}{15} = £\frac{1}{15} \times 12 = 1\text{s. 4d.} \times 12 = 16\text{s.}$$

$$£\frac{4}{5} = £\frac{1}{5} \times 4 = 4\text{s.} \times 4 = 16\text{s.}$$

In calculations involving ordinary fractions, it is advisable first to reduce such fractions, when necessary, to their lowest terms.

Ex. 4. Reduce each of the following fractions to its lowest terms :

$$(i) \frac{462}{539}; \quad (ii) \frac{897}{1173}.$$

(i) Resolve both numerator and denominator into prime factors, i.e. factors which cannot be further resolved ; thus

$$462 = 2 \times 231 = 2 \times 3 \times 77 = 2 \times 3 \times 7 \times 11,$$

$$539 = 7 \times 77 = 7 \times 7 \times 11 ;$$

$$\therefore \frac{462}{539} = \frac{2 \times 3 \times 7 \times 11}{7 \times 7 \times 11} = \frac{2 \times 3}{7} = \frac{6}{7},$$

on cancelling the common factors 7 and 11.

(ii) In the same way,

$$897 = 3 \times 299 = 3 \times 13 \times 23,$$

$$1173 = 3 \times 391 = 3 \times 17 \times 23.$$

$$\therefore \frac{897}{1173} = \frac{3 \times 13 \times 23}{3 \times 17 \times 23} = \frac{13}{17}.$$

EXERCISE 1A

1. A traveller received 96 rupees in exchange for £7 3s. 6d. ; find the value of a rupee in pence and a fraction of a penny.

2. If the equivalent of £28 in French money is 5005 francs, find the rate of exchange, i.e. the number of francs in £1 at the time.

3. Find, to the nearest penny, the values of :

$$(i) \frac{9}{13} \text{ of } £1, \quad (ii) \frac{5}{8} \text{ of } £5 \text{ 3s. 7d.}$$

4. Calculate, to the nearest lb., the values of :

$$(i) \frac{7}{17} \text{ of } 1 \text{ ton}, \quad (ii) \frac{16}{19} \text{ of } 4 \text{ tons } 7 \text{ cwt.}$$

5. What fraction is (i) £1 5s. 8d. of £7, (ii) 6 cwt. 25 lb. of 8 cwt. 1 qr. 19 lb.?

6. Find what fraction 4s. 7d. is of £1 ; hence calculate, to the nearest penny, the dividend on £13 12s. 3d. at 4s. 7d. in the £.

7. Calculate, to the nearest penny, the import duty on 82 cwt. of sugar at 6s. 6 $\frac{3}{4}$ d. per cwt.

8. Express 4 cwt. 1 qr. 5 lb. as a fraction of 6 cwt. 1 qr. 3 lb. Hence, find the price of a case of goods weighing 4 cwt. 1 qr. 5 lb. if the cost of a case of similar goods weighing 6 cwt. 1 qr. 3 lb. is £3 17s. 7d.

9. Reduce each of the following fractions to its lowest terms :

$$(i) \frac{134}{469}, \quad (ii) \frac{429}{693}, \quad (iii) \frac{759}{1173}.$$

10. The accounts of Smith & Son for the year were as follows :

	£
Purchase of goods for sale - - -	11,082
Wages and salaries - - -	9,306
Rent, rates, lighting and heating -	643
Cleaning and general expenses -	848
Sales - - - - -	26,741

Calculate the profit as a fraction, in its lowest terms, of the total expenditure.

11. When the exchange rate between London and New York is $4\frac{4}{5}$ dollars to the £, calculate the number of dollars equivalent to £51 9s. 2d.

12. Dress material bought in Paris at a certain cost was imported to London, the duty, carriage and insurance amounting to $\frac{1}{5}$ of the prime cost. It was sold in London for £12; if the rate of exchange was $178\frac{3}{4}$ francs to the £, calculate the cost at which the material was bought in Paris.

13. A trader buys (i) 171 articles for £2108 11s. and sells them for 13 guineas each, (ii) 228 articles at £6 4s. 3d. each and sells them for £1613 17s.

Calculate his profit as a fraction of the total outlay.

14. A, B and C enter into a business partnership. A contributes a capital of £5852, B $\frac{4}{7}$ of A's capital and C $\frac{21}{44}$ of B's capital. At the end of the year, a profit was declared of £2272; find the amounts contributed by B and C and express the profit as a fraction of the total capital.

15. A man's earned income is £485 per annum. He pays income tax on four-fifths of all over £180, the rate being 1s. 8d. in the £ on the first £135 and 5s. 6d. in the £ on the remainder. What fraction of his total income does he pay in tax?

1·5. Classification of Fractions.

When the numerator of a fraction is less than the denominator, the fraction is said to be **Proper**; thus $\frac{8}{11}$, $\frac{23}{37}$, $\frac{113}{115}$ are proper fractions. Each of these fractions represents a part less than the whole unit.

When the numerator is greater than the denominator, the fraction is said to be **Improper**; thus $\frac{19}{8}$, $\frac{41}{13}$, $\frac{237}{41}$ are improper fractions. These can be expressed as whole numbers and proper fractions, thus

$$\frac{19}{8} = \frac{16+3}{8} = \frac{16}{8} + \frac{3}{8} = 2 + \frac{3}{8}.$$

The plus sign is not generally used, so that $2 + \frac{3}{8}$ is written $2\frac{3}{8}$.

Similarly $\frac{41}{13} = 3\frac{2}{13}$ and $\frac{237}{41} = 5\frac{32}{41}$.

Improper fractions expressed in this form are called **Mixed Numbers**; hence $2\frac{3}{8}$, $3\frac{2}{13}$, $5\frac{32}{41}$ are mixed numbers.

Hence, to convert an improper fraction into a mixed number, divide the numerator by the denominator and express the remainder as a fraction of the denominator.

1·6. Multiplication of Fractions.

The multiplication of a fraction by a whole number has already been effected several times in Exercises 2 and 3, where it will be seen that the numerator is multiplied by the whole number whilst the denominator remains unaltered.

To multiply a fraction by another fraction is also quite straightforward. Consider, for instance, the product $\frac{2}{3} \times \frac{4}{5}$. Suppose this fraction of one hour were required: first find the value of $\frac{4}{5}$ of 1 hour, multiply it by $\frac{2}{3}$ and express the result as a fraction of 1 hour.

$$\frac{4}{5} \text{ of 1 hour} = \frac{4}{5} \text{ of 60 minutes} = \frac{4 \times 60}{5} \text{ min.} = 48 \text{ min.}$$

$$\text{and } \frac{2}{3} \text{ of 48 min.} = \frac{2 \times 48}{3} \text{ min.} = 32 \text{ min.} = \frac{32}{60} \text{ of 1 hour}$$

$$= \frac{8 \times 4}{15 \times 4} \text{ hr.} = \frac{8}{15} \text{ hr.};$$

$$\therefore \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}, \text{ i.e. } \frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

Hence the product of two fractions is a fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators.

When mixed numbers occur, they must first be expressed in improper fractions; for example,

$$2\frac{1}{4} \times \frac{3}{7} = \frac{9}{4} \times \frac{3}{7} = \frac{9 \times 3}{4 \times 7} = \frac{27}{28},$$

$$3\frac{1}{5} \times 2\frac{1}{3} = \frac{16}{5} \times \frac{7}{3} = \frac{16 \times 7}{5 \times 3} = \frac{112}{15} = 7\frac{7}{15}.$$

Sometimes, common factors may be cancelled out, as in Ex. 5.

Ex. 5. *The volume of a rectangular solid is found by multiplying together its length, breadth and depth. Find the volume of a solid whose length is $3\frac{3}{4}$ feet, breadth $2\frac{2}{3}$ ft. and depth $1\frac{1}{4}$ ft.*

Evidently the volume in cubic feet is

$$\begin{aligned} 3\frac{3}{4} \times 2\frac{2}{3} \times 1\frac{1}{4} &= \frac{15}{4} \times \frac{8}{3} \times \frac{5}{4} = \frac{5 \times \cancel{3} \times \cancel{4} \times 2 \times 5}{2 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \\ &= \frac{5 \times 5}{2} = \frac{25}{2} = 12\frac{1}{2}. \end{aligned}$$

1.7. A Fraction of a Fraction.

When the word “of” occurs between two fractions, it indicates multiplication; for example:

$$(i) \frac{2}{3} \text{ of } 12 = \frac{2}{3} \times 12 = \frac{2 \times 12}{3} = 8. \quad (ii) \frac{2}{7} \text{ of } \frac{5}{9} = \frac{2}{7} \times \frac{5}{9} = \frac{10}{63}.$$

$$(iii) \frac{4}{13} \text{ of } 4\frac{1}{16} = \frac{4}{13} \times 4\frac{1}{16} = \frac{\cancel{4}}{\cancel{13}} \times \frac{\cancel{4}^5}{\cancel{16}_{\cancel{4}}} = \frac{5}{13} = 1\frac{1}{13}.$$

Hence, in working, the word *of* must always be replaced by the sign \times .

1·8. Division of Fractions.

Consider the following simple problems in division.

$$(i) \quad 6 \div \frac{1}{4}.$$

Since 1 unit contains four *fourths*, it is evident that 6 units will contain 6×4 or 24 *fourths* ;
i.e. $6 \div \frac{1}{4} = 24.$

$$(ii) \quad \frac{2}{3} \div 5.$$

Now one unit divided by 5 is written $\frac{1}{5}$, so that $\frac{1}{3}$ of a unit divided by 5 is $\frac{1}{3}$ of $\frac{1}{5}$ or $\frac{1}{15}$.

$$\therefore \frac{2}{3} \text{ of a unit divided by } 5 = \frac{1}{15} \times 2 = \frac{2}{15},$$

$$\text{i.e.} \quad \frac{2}{3} \div 5 = \frac{2}{15}.$$

$$(iii) \quad \frac{4}{7} \div \frac{3}{5}.$$

Since one unit contains five *fifths*, therefore $\frac{4}{7}$ of a unit will contain $\frac{4}{7} \times 5$ or $\frac{20}{7}$ *fifths*.

But the divisor is $\frac{3}{5}$, so that this result will be three times too large ; hence $\frac{4}{7} \div \frac{3}{5} = \frac{20}{7} \times \frac{1}{3} = \frac{20}{21}.$

Note from these examples that

$$6 \div \frac{1}{4} = 24 = 6 \times \frac{4}{1}, \quad \frac{2}{3} \div 5 = \frac{2}{15} = \frac{2}{3} \times \frac{1}{5}, \\ \frac{4}{7} \div \frac{3}{5} = \frac{20}{21} = \frac{4}{7} \times \frac{5}{3}.$$

Hence, in each case the result may be obtained by inverting the divisor and multiplying ; this is true generally and the rule for the division of fractions may be stated as follows :

To divide any number, whether fractional or not, by a fraction, invert the divisor and multiply.

Ex. 6. (i) Divide $2\frac{13}{21}$ by $3\frac{1}{7}$.

(ii) 4 tons 9 cwt. of building material cost £20 15s. 4d. ; calculate the cost per ton.

$$(i) \quad 2\frac{13}{21} \div 3\frac{1}{7} = \frac{55}{21} \div \frac{22}{7} = \frac{55}{21} \times \frac{7}{22} = \frac{5}{6}.$$

(ii) First express 9 cwt. as a fraction of a ton and 15s. 4d. as a fraction of a £.

Now 9 cwt. = $\frac{9}{20}$ of a ton, so that 4 tons 9 cwt. = $4\frac{9}{20}$ tons.

Also 15s. 4d. = 46 fourpences, and since there are 60 fourpences in £1,

$$15s. 4d. = \frac{46}{60} \text{ of } £1 = £\frac{23}{30},$$

$$\therefore £20 \ 15s. 4d. = £20\frac{23}{30}.$$

Hence, since $£20\frac{23}{30}$ is the price of $4\frac{9}{20}$ tons,

$$\therefore \text{cost per ton} = £(20\frac{23}{30} \div 4\frac{9}{20}) = £(\frac{623}{30} \div \frac{89}{20})$$

$$= £\left(\frac{\overset{7}{\cancel{623}}}{\underset{3}{\cancel{30}}} \times \frac{\overset{2}{\cancel{20}}}{\cancel{89}}\right) = £\frac{14}{3} = £4\frac{2}{3}$$

$$= £4 \ 13s. 4d.$$

1.9. Addition and Subtraction of Fractions.

Suppose it is required to find the value of $\frac{1}{6} + \frac{2}{5} + \frac{3}{8}$.

Consider a concrete case by finding what fraction of £1 the above fraction represents.

$$\text{Now } £\frac{1}{6} = \frac{240}{6} \text{ pence} = 40 \text{ pence,}$$

$$£\frac{2}{5} = \frac{2 \times 240}{5} \text{ pence} = 2 \times 48 \text{ pence} = 96 \text{ pence,}$$

$$\text{and } £\frac{3}{8} = \frac{3 \times 240}{8} \text{ pence} = 3 \times 30 \text{ pence} = 90 \text{ pence.}$$

$$\begin{aligned} \therefore £\left(\frac{1}{6} + \frac{2}{5} + \frac{3}{8}\right) &= (40 + 96 + 90) \text{ pence} = 226 \text{ pence} \\ &= £\frac{226}{240} = £\frac{113}{120}. \end{aligned}$$

$$\therefore \frac{1}{6} + \frac{2}{5} + \frac{3}{8} = \frac{113}{120}.$$

To obtain this result, £1 was chosen because each of the fractions of £1 can be expressed exactly in pence ; addition is then possible because each fraction has been expressed in terms of the *same unit*. This is the basic principle in adding and subtracting fractions.

From the above example, it will be seen that 120 is the *least* number which contains 6, 5 and 8 exactly ; it is called the **Least Common Multiple (L.C.M.)** of the three denominators and each fraction may be expressed with 120 as its denominator, for

$$\frac{1}{6} = \frac{1 \times 20}{6 \times 20} = \frac{20}{120} ; \quad \frac{2}{5} = \frac{2 \times 24}{5 \times 24} = \frac{48}{120} ; \quad \frac{3}{8} = \frac{3 \times 15}{8 \times 15} = \frac{45}{120},$$

$$\frac{1}{6} + \frac{2}{5} + \frac{3}{8} = \frac{20}{120} + \frac{48}{120} + \frac{45}{120} = \frac{113}{120}.$$

The same principle will apply to the subtraction of fractions ; hence, to add or subtract fractions, all the fractions must be expressed with the **same denominator** which, in order to reduce the working as much as possible, should be the **Least Common Multiple** of the given denominators.

Ex. 7. *Simplify* (i) $\frac{4}{7} + \frac{3}{4} - \frac{5}{8}$, (ii) $4\frac{1}{5} + 3\frac{5}{8} - 5\frac{1}{3}$.

(i) The L.C.M. of 7, 4, 8 = L.C.M. of 7 and 8, since 4 is contained in 8,

$$= 7 \times 8 = 56.$$

Hence, each fraction must be expressed with a denominator 56 ;

$$\therefore \frac{4}{7} + \frac{3}{4} - \frac{5}{8} = \frac{32}{56} + \frac{42}{56} - \frac{35}{56} = \frac{39}{56}.$$

(ii) There is no need to convert the mixed numbers into improper fractions ; deal with the whole numbers and the proper fractions separately.

The L.C.M. of 5, 6, 3 = L.C.M. of 5 and 6 = $5 \times 6 = 30$.

$$\begin{aligned} \therefore 4\frac{1}{5} + 3\frac{5}{6} - 5\frac{1}{3} &= 4 + 3 - 5 + \frac{1}{5} + \frac{5}{6} - \frac{1}{3} \\ &= 2 + \frac{6}{30} + \frac{25}{30} - \frac{10}{30} = 2 + \frac{21}{30} \\ &= 2 + \frac{7}{10} = 2\frac{7}{10}. \end{aligned}$$

When the proper fraction to be subtracted is greater than that from which it has to be taken, the following method may be used.

Ex. 8. Subtract $2\frac{2}{3}$ from $4\frac{4}{7}$.

$$\begin{aligned} 4\frac{4}{7} - 2\frac{2}{3} &= 4 - 2 + \frac{4}{7} - \frac{2}{3} = 2 + \frac{12}{21} - \frac{14}{21} \\ &= 1 + \frac{21}{21} + \frac{12}{21} - \frac{14}{21} = 1 + \frac{19}{21} = 1\frac{19}{21}. \end{aligned}$$

Those familiar with negative numbers could proceed as follows :

$$2 + \frac{12}{21} - \frac{14}{21} = 2 - \frac{2}{21} = 1 + \frac{19}{21} = 1\frac{19}{21}.$$

1·10. Simplification of Mixed Fractions.

In the solution of some practical problems, it is occasionally necessary to use mixed or complex fractions in which a number of simple fractions are connected by the signs $+$, $-$, \times , \div and the word *of*. The only difficulty that may arise in simplifying such fractions lies in the order in which the various operations should be taken.

Consider a simple case with whole numbers. What is the value of $8 + 7 \times 3$? Is it $8 + 21 = 29$, or $(8 + 7) \times 3 = 15 \times 3 = 45$? The former is the correct answer because multiplication is merely a shortened form of addition ; thus

$$8 + 7 \times 3 = 8 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 8 + 21 = 29.$$

It is clear, therefore, that multiplication must be carried out first. Since the rule for division is to invert the divisor and multiply, division must follow the same order as multiplication.

Hence : In dealing with mixed fractions, the word “*of*” implies multiplication, and all multiplication and division must first be carried out before addition and subtraction, unless the presence of brackets indicates a different order.

The following examples will illustrate the application of this rule.

Ex. 9. Simplify $\frac{1}{6} + \frac{3}{5}$ of $\frac{8}{9} - \frac{5}{12} \div 4\frac{1}{6}$.

Remembering that *of* means multiplication, which must be carried out first,

$$\frac{1}{6} + \frac{3}{5} \text{ of } \frac{8}{9} - \frac{5}{12} \div 4\frac{1}{6} = \frac{1}{6} + \frac{3}{5} \times \frac{8}{9} - \frac{5}{12} \div \frac{25}{6}$$

3

$$= \frac{1}{6} + \frac{8}{15} - \frac{\cancel{5}}{\cancel{12}} \times \frac{\cancel{6}}{\cancel{25}} = \frac{1}{6} + \frac{8}{15} - \frac{1}{10} = \frac{5}{30} + \frac{16}{30} - \frac{3}{30} = \frac{18}{30} = \frac{3}{5}.$$

Ex. 10. Simplify $\frac{\frac{4}{5} + \frac{33}{52} \div 1\frac{9}{13}}{2\frac{1}{5} + \frac{2}{3} \text{ of } (\frac{5}{8} - \frac{2}{5})}$.

This is a more complicated example, for numerator and denominator are both fractions, but as a fraction really denotes division, the given fraction

$$\begin{aligned} &= \left(\frac{4}{5} + \frac{33}{52} \div 1\frac{9}{13} \right) \div \left\{ 2\frac{1}{5} + \frac{2}{3} \text{ of } \left(\frac{5}{8} - \frac{2}{5} \right) \right\} \\ &= \left(\frac{4}{5} + \frac{33}{52} \div \frac{22}{13} \right) \div \left\{ 2\frac{1}{5} + \frac{2}{3} \times \left(\frac{25}{40} - \frac{16}{40} \right) \right\} \\ &= \left(\frac{4}{5} + \frac{33}{52} \times \frac{13}{22} \right) \div \left(2\frac{1}{5} + \frac{2}{3} \times \frac{9}{40} \right) = \left(\frac{4}{5} + \frac{3}{8} \right) \div \left(2\frac{1}{5} + \frac{3}{20} \right) \\ &= \left(\frac{32}{40} + \frac{15}{40} \right) \div \left(2 + \frac{4}{20} + \frac{3}{20} \right) = \frac{47}{40} \div 2\frac{7}{20} = \frac{47}{40} \div \frac{47}{20} \\ &= \frac{47}{40} \times \frac{20}{47} = \frac{1}{2}. \end{aligned}$$

1.11. Simple Problems.

The chief use of ordinary fractions is to solve practical problems, and the following examples will illustrate the method of application in simple cases. More difficult problems will be considered later.

Ex. 11. Convert a price of 5s. 4d. per gallon into francs per litre, given that £1 = 178½ francs and 1 litre = 1¼ pints. Give the result to the nearest half-franc.

First convert the price to francs :

$$\begin{aligned} 5\text{s. } 4\text{d.} &= \text{£} \frac{64}{40} = \frac{64}{40} \times 178\frac{1}{2} \text{ francs} = \frac{64}{240} \times \frac{357}{2} \text{ francs} \\ &= \frac{238}{5} \text{ francs.} \end{aligned}$$

Now convert gallons into litres :

$$\begin{aligned} 1 \text{ gallon} &= 8 \text{ pints} = 8 \div 1\frac{3}{4} \text{ litres} = 8 \div \frac{7}{4} \text{ litres} \\ &= 8 \times \frac{4}{7} \text{ litres} = \frac{32}{7} \text{ litres.} \end{aligned}$$

$$\therefore \text{ cost of } \frac{32}{7} \text{ litres} = \frac{238}{5} \text{ francs,}$$

so that the price of 1 litre = $\frac{238}{5} \div \frac{32}{7}$ francs

$$= \frac{238}{5} \times \frac{7}{32} \text{ francs}$$

$$= \frac{833}{80} \text{ francs} = 10\frac{33}{80} \text{ francs.}$$

Now $\frac{33}{80}$ is less than $\frac{1}{2}$ by $\frac{7}{80}$, so that $10\frac{33}{80}$ is nearer $10\frac{1}{2}$ than 10. Hence, to the nearest half-franc, the required price is

$10\frac{1}{2}$ francs per litre.

Ex. 12. *A sovereign contains $\frac{11}{12}$ fine gold by weight and 1869 sovereigns weigh 480 oz. Troy. Calculate, to the nearest penny, the value of a sovereign when gold is £6 13s. 6d. per oz. Troy.*

$$\text{Weight of one sovereign} = \frac{480}{1869} \text{ oz.}$$

$$\therefore \text{weight of gold in one sovereign} = \frac{480}{1869} \times \frac{11}{12} \text{ oz.}$$

$$\text{Since } £6 \text{ 13s. 6d.} = 133\frac{1}{2} \text{ shillings,}$$

$$\text{value of one sovereign in shillings} = \frac{480}{1869} \times \frac{11}{12} \times 133\frac{1}{2}$$

$$\begin{aligned} &= \frac{20}{7} \times \frac{480}{1869} \times \frac{11}{12} \times \frac{267}{2} = \frac{220}{7} = 31\frac{3}{7}. \end{aligned}$$

$$\text{Now } \frac{3}{7} \text{ of a shilling} = \frac{3}{7} \times 12 \text{ pence} = \frac{36}{7} \text{ pence} = 5\frac{1}{7} \text{ pence.}$$

Hence, the value of a sovereign, to the nearest penny, is 31s. 5d.,

i.e.

£1 11s. 5d.

EXERCISE 1B

1. Simplify $6\frac{1}{4} \times 4\frac{2}{5} \times 3\frac{1}{3} \div (5\frac{5}{7} \times 1\frac{3}{4})$. 46

2. Find, to the nearest penny, the value of

$$£\{(1\frac{7}{15} \times \frac{5}{8}) - (2\frac{1}{21} \div 4\frac{1}{3})\}.$$

3. What fraction of a kilogram is 1 lb. 6 oz., taking 1 kilogram to be equivalent to $2\frac{1}{5}$ lb.? 79

Express 4 cwt. 3 qr. 7 lb. in kilograms.

4. Express as a fraction of one shilling the difference between $\frac{9}{34}$ of 9s. 11d. and $\frac{7}{26}$ of 6s. 6d. (U.L.C.I.)

5. Express a speed of 28 knots in miles per hour to the nearest quarter of a mile, given that 1 knot = a speed of 1 nautical mile per hour and 1 nautical mile = 6080 feet.

6. Convert $\text{£}91\frac{1}{5}$ German Marks into £ s when the exchange rate is $12\frac{4}{5}$ Marks to $\text{£}1$. $\text{£}7\ 10\text{/-}$

7. Taking 5 metres to be equivalent to $196\frac{4}{5}$ inches, calculate, to the nearest mile, the equivalent of 8 kilometres in miles.

1 kilometre = 1000 metres. $5\frac{1}{2}$

8. Express $793\frac{4}{5}$ kilograms as a fraction of a ton, having given that 1 kilogram = 1000 grams and 1 lb. = $453\frac{3}{5}$ grams.

9. Express (i) 17s. 3d. as a fraction of $\text{£}1$, and

(ii) 8 cwt. 2 qr. 24 lb. as a fraction of 1 ton. $\frac{1}{140}$

Hence, calculate the price per ton of some material of which the charge for 13 tons 8 cwt. 2 qr. 24 lb. is $\text{£}54\ 17\text{s. } 3\text{d.}$

10. Express $\text{£}5\ 14\text{s. } 8\text{d.}$ as a fraction of $\text{£}7\ 10\text{s. } 6\text{d.}$; hence find what weight of goods can be purchased for $\text{£}5\ 14\text{s. } 8\text{d.}$ if $5\frac{1}{4}$ tons of similar goods cost $\text{£}7\ 10\text{s. } 6\text{d.}$

11. By expressing 8 cwt. 3 qr. 20 lb. as a fraction of a ton, calculate the cost of 8 cwt. 3 qr. 20 lb. at $\text{£}10\ 14\text{s. } 8\text{d.}$ per ton.

12. Express 9 cwt. 3 qr. 8 lb. as a fraction of 1 ton; hence convert 52 tons 9 cwt. 3 qr. 8 lb. into metric tons, having given that 1 metric ton = $2204\frac{5}{8}$ lb.

13. The cost of 83 tons 17 cwt. 64 lb. of material is $\text{£}2152\ 17\text{s. } 8\text{d.}$; find the average cost per ton.

14. Find the cost per ton of material for which $\text{£}75\ 17\text{s. } 3\text{d.}$ is paid for 13 tons 7 cwt. 3 qr.

15. 74 cwt. 3 qr. 26 lb. of a commodity cost $\text{£}16\ 3\text{s.}$; what is the cost of 7 cwt. 9 lb. at the same rate?

16. If 5 tons 12 cwt. 14 lb. cost $\text{£}183\ 2\text{s. } 9\text{d.}$, calculate the average cost per ton.

17. Simplify $\frac{41}{91} + \frac{2}{247} - \frac{4}{133}$, giving the result in its lowest terms. $\frac{3}{21}$

18. Find, in its lowest terms, the simplified value of $\frac{13}{31} + \frac{2}{33} - \frac{4}{99}$.

19. Evaluate (i) $\frac{2}{3} + \frac{5}{6} \times \frac{7}{10}$, (ii) $(\frac{2}{3} + \frac{5}{6}) \div 4\frac{1}{2}$. (R.S.A.)

20. The rates payable by the ratepayers of a certain town are as follows:

Education, Health, Lighting and Cleaning - $49\frac{3}{4}\text{d.}$ in the £ .

Maintenance of the Police, etc. - - - $5\frac{9}{25}\text{d.}$ in the £ .

County Rate - - - - - $38\frac{13}{20}\text{d.}$ in the £ .

Towards these, the town receives a grant of $17\frac{19}{25}\text{d.}$ in the £ from the Government. How much in the £ must each ratepayer pay?

21. Simplify $\frac{(\frac{1}{2} + \frac{1}{4}) \div (\frac{5}{6} \text{ of } \frac{3}{8})}{2\frac{2}{3} \div (3\frac{1}{3} - 2\frac{1}{2})}$. (U.L.C.I.)

22. Simplify $\frac{2\frac{1}{3} - (\frac{4}{5} \times 2\frac{1}{2}) \cdot 1\frac{2}{3}}{(\frac{1}{6} \times 3\frac{2}{5}) + 1\frac{1}{2} \div 1\frac{4}{17}}$. (U.L.C.I.)

23. Of a man's weekly wage, $\frac{1}{5}$ was spent on rent and $\frac{2}{3}$ of the remainder on food. $\frac{1}{2}$ of the amount remaining was needed for various expenses and he was left with 10s. What was his weekly wage? $\pounds 3 \text{ } 15\text{ } -$ (L.Ch.C.)

24. A firm buys 26 tons of a commodity at £5 11s. 9d. per ton, and then a further 42 tons at £4 19s. 4d. per ton and finally 81 tons at £6 4s. 2d. per ton. At what average price per ton must the whole be sold to ensure a profit of one shilling in the £ on the outlay?

25. The profits of a business for five consecutive years were :

£5631, £5278, £4488, £4931, £5852.

Express the average profit as a fraction in its lowest terms of

(i) the best year's profit, $\frac{1}{19}$

(ii) the worst year's profit. $\frac{1}{16}$

26. A's house is assessed at £42 10s. and he pays rates for the year at 11s. 4 $\frac{2}{5}$ d. in the £. B, living in another town, pays rates for the year at 12s. 10 $\frac{1}{4}$ d. in the £ and his house is assessed at £37 10s. Who pays the greater amount and by how much?

27. At what price must a jeweller sell an 18-carat gold chain, weighing 2 $\frac{2}{5}$ oz. Troy, when gold is worth 142 shillings per oz. Troy, the price to include a profit of 5s. in the £ on the cost price? Fine gold is 24 carat.

28. In France goods are purchased at 19 $\frac{1}{2}$ francs per kilogram and sent to London for sale. Duty and carriage have to be paid at the rate of one-third of the prime cost. Find the price per lb., to the nearest halfpenny, at which the goods must be sold in London so that a profit of 2s. in the £ may be made on the outlay, given that £1 = 178 $\frac{3}{4}$ francs and 1 kilogram = 2 $\frac{1}{5}$ lb.

29. A barrel of butter containing 100 kilograms is bought for 264 kronen in Denmark, including the cost of carriage to England. Import duty at the rate of 14s. per cwt. is paid and the butter is sold at a profit of 4s. in the £ on the total cost. Calculate the selling price per lb., to the nearest penny, given that 1 kilogram = 2 $\frac{1}{5}$ lb. and £1 = 22 $\frac{2}{5}$ kronen.

30. On June 1st a man had £32 10s. in the Post Office Savings Bank. During the next few months he made the following deposits :

July 19	-	-	-	-	£2 10s.
October 21	-	-	-	-	£3 5s.
November 13	-	-	-	-	£4.

On August 31st he withdrew £5 10s. Calculate the amount of interest due to him for the six months ending on December 31st, the rate of interest being 6d. per year for each complete pound deposited for a complete month and beginning on the first day of the month following the deposit. Each month is to be taken as one-twelfth of a year.

Express the interest as a fraction of the total deposit standing to his credit on December 31st.

CHAPTER II

DECIMAL FRACTIONS

2.1. The Index Notation.

The product of any number of equal factors may be expressed very briefly by means of the *index notation*. If, for example, a stands for any number, then

$a \times a$ is written a^2 and is read " a to the *second power*" or " a *squared*";

$a \times a \times a$ is written a^3 and is read " a to the *third power*" or " a *cubed*";

$a \times a \times a \times a \times a \times a \times a$ is written a^7 and is read " a to the *seventh power*",

and so on.

The small figure denotes the number of equal factors and is called an *index*.

The second power is often called the *square* and the third power the *cube* of the number concerned; thus

the square of $13 = 13^2 = 13 \times 13 = 169$,

the cube of $7 = 7^3 = 7 \times 7 \times 7 = 343$.

Similarly, $100 = 10^2$; $1000 = 10^3$; $1,000,000 = 10^6$.

In the case of a power of 10, note that the index shows the number of noughts; thus, one million has six noughts and is represented by 10^6 .

2.2. The Decimal Notation.

The reading of any number gives a clue to its formation. For example, seven *thousand* four *hundred* and *eighty-three* is written 7483 and means

$$(7 \times 1000) + (4 \times 100) + (8 \times 10) + 3,$$

i.e.
$$(7 \times 10^3) + (4 \times 10^2) + (8 \times 10) + 3.$$

Similarly, any other number may be expressed in this form, so that the universal system of expressing numbers is based upon 10 and powers of 10; it is therefore known as the **decimal notation**, from the Latin word *decem* meaning *ten*, as in *December*, which was the name of the *tenth* month in the early Roman year.

Now take a number like 3333 containing the same digits. Writing it in the table, shown on the right, it will be clear that the 3 in the third place has only one-tenth of the value of the 3 in the fourth place, and the 3 in the second place has only one-tenth the value of the 3 in the third place, and so on. Hence, in moving from the left to the right, each digit has one-tenth the value of the digit next to it on the left.

THOUSANDS (Fourth Place)	HUNDREDS (Third Place)	TENS (Second Place)	UNITS (First Place)
3	3	3	3

But there is no need to stop at the units' place. The same system of notation can be extended to the right of the units' place. The first figure to the right would be one-tenth of a unit, the next one-hundredth, the next one-thousandth, and so on for any number of places.

2·3. A Natural Extension.

Suppose a surveyor measures the length of a strip of ground and finds it to be 7 chains 77 links. Since the surveyor's chain contains 100 links, he could write his measurement down as $7\frac{77}{100}$ chains.

$$\text{But } \frac{77}{100} = \frac{70+7}{100} = \frac{7}{10} + \frac{7}{100} = \frac{7}{10} + \frac{7}{10^2},$$

so that, if we could extend the above rule to digits written to the right of the unit, $7\frac{77}{100}$ could be written as $7\cdot77$, where the dash indicates the unit. Instead of the dash, it is more usual to write a dot in the middle of the line, thus, $7\cdot77$. This dot is called the **decimal point**, and, since $0\cdot77 = \frac{77}{100}$ and is therefore a fraction, $0\cdot77$ is known as a **decimal fraction**.

A decimal fraction is therefore a particular form of an ordinary or vulgar fraction whose denominator is 10 or a power of 10, but which is not written.

The following table shows the composition of a number beginning with thousands and ending with *thousandths*. The table can easily be extended in either direction to tens of thousands and *ten-thousandths*; hundreds of thousands and *hundred-thousandths*, etc.

The Place Values -	10^3	10^2	10	1	The DECIMAL POINT	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$
	Thousands	Hundreds	Tens	UNITS		<i>Tenths</i>	<i>Hundredths</i>	<i>Thousandths</i>
	8	3	2	7		4	2	8
Any Number -	Whole Number					Fraction		

Thus, the number 8327.428 means

$(8 \times 10^3) + (3 \times 10^2) + (2 \times 10) + 7 + \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3}$, and must be read,

Eight thousand three hundred and twenty-seven *point four two eight*.

It is interesting to know that this extension of the decimal notation to include fractions was first introduced in 1585 by *Stevinus*, a Dutchman, who wrote 35 8'7''6''' for 35.876. It was not until the eighteenth century that the use of the dot became general.

Care must be taken not to confuse 6.7 with 6 . 7, where the dot is written between two numbers in the same position as a full stop. 6.7 means $6\frac{7}{10}$, but 6 . 7 means 6×7 .

2.4. Conversion of a Decimal into an Ordinary Fraction.

Decimal fractions may readily be expressed as ordinary fractions from their definition.

Ex. 1. Convert 0.25, 0.925, 0.096 and 3.4375 into ordinary fractions.

$$0.25 = \frac{2}{10} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}.$$

$$0.925 = \frac{9}{10} + \frac{2}{100} + \frac{5}{1000} = \frac{925}{1000} = \frac{37}{40}.$$

$$0.096 = \frac{0}{10} + \frac{9}{100} + \frac{6}{1000} = \frac{96}{1000} = \frac{12}{125}.$$

$$\begin{aligned} 3.4375 &= 3 + \frac{4}{10} + \frac{3}{100} + \frac{7}{1000} + \frac{5}{10000} = 3 + \frac{4375}{10000} \\ &= 3 + \frac{7}{16} = 3\frac{7}{16}. \end{aligned}$$

From these results, it should be observed that the numerator of the equivalent ordinary fraction is formed by the figures of the decimal and the denominator is 1 followed by as many noughts as there are decimal places, in the fraction ; thus

$$0.9 = \frac{9}{10}, \quad 0.97 = \frac{97}{100}, \quad 0.063 = \frac{63}{1000},$$

and so on.

Note carefully the difference between 0.90 and 0.09 ;

$$0.90 = \frac{9}{10} + \frac{0}{10^2} = \frac{9}{10} = 0.9, \quad \text{but} \quad 0.09 = \frac{0}{10} + \frac{9}{10^2} = \frac{9}{100};$$

hence, it will be evident that it is useless to write noughts at the end of a decimal ; 0.90 is the same as 0.9. Where a zero is followed, however, by a digit on the right, that zero then counts as a digit, for we have just seen that $0.09 = \frac{9}{100}$; similarly, $0.307 = \frac{307}{1000}$, and so on.

Ex. 2. Express (i) £6.075 in £ s. d., and (ii) 0.0625 cwt. in lb.

$$(i) \quad £6.075 = £6\frac{75}{1000} = £6\frac{3}{40} = £6 \text{ 1s. 6d.}$$

$$(ii) \quad 0.0625 \text{ cwt.} = \frac{625}{10000} \text{ cwt.} = \frac{5}{80} \text{ cwt.} = \frac{1}{16} \text{ cwt.} = 7 \text{ lb.}$$

2.5. Conversion of an Ordinary Fraction into a Decimal.

Any ordinary or vulgar fraction may be expressed as a decimal by carrying the division implied to the right of the unit by means of the extended decimal notation. For example,

$$\frac{3}{4} = \frac{3 \times 100}{4 \times 100} = \frac{300}{4} \times \frac{1}{100} = \frac{75}{100} = 0.75.$$

The working may be shown briefly as follows :

$$4 \overline{) 3.00} \quad \therefore \frac{3}{4} = 0.75.$$

Ex. 3. Convert $\frac{5}{8}$, $\frac{7}{16}$ and $3\frac{23}{40}$ into decimals.

$$\frac{5}{8} = 5 \div 8 = 5.000 \div 8 = 0.625.$$

Similarly,

$$\frac{7}{16} = 7 \div 16.$$

Carrying out the division,

$$\frac{7}{16} = 0.4375.$$

$$\begin{array}{r} 4 \overline{) 7.0000} \\ 4 \overline{) 1.7500} \\ \hline 0.4375 \end{array}$$

And

$$\frac{23}{40} = \frac{2.3}{4} = \frac{2.300}{4} = 0.575 ;$$

$$\therefore 3\frac{23}{40} = 3.575.$$

2.6. Inexact Decimal Equivalents.

The expression of an ordinary fraction as a decimal depends really upon converting the denominator into a *power* of ten. As, however, the prime factors of 10 are 2 and 5, it will not be possible to find the required power of 10 when the denominator of the ordinary fraction contains factors other than 2 or 5. Thus,

$$\frac{1}{3} = 1 \div 3, \quad \text{i.e. } 3 \overline{) 1.0000 \dots} \\ \underline{0.3333 \dots}$$

$\therefore \frac{1}{3} = 0.3333 \dots$ the threes being continued indefinitely.

Again,

$$\frac{1}{7} = 1 \div 7, \quad \text{i.e. } 7 \overline{) 1.00000000 \dots} \\ \underline{0.14285714 \dots}$$

Note that when 7 is reached in the quotient, there is a remainder of 1, so that the quotient repeats itself on further division ;

$$\therefore \frac{1}{7} = 0.14285714285714 \dots,$$

the figures 142857 repeating themselves indefinitely.

In commercial and technical calculations, it is unnecessary to use more decimal places than are required to give an accurately

practical result and, as a consequence, approximations have to be made. In obtaining approximate answers, however, the fact to be remembered is that they must be correct *as far as they are stated*. In general, an approximate answer may be described in either of the following forms :

(i) correct to n decimal places, or

(ii) correct to n significant figures,

where n denotes any positive integer.

These are illustrated in the following examples.

Ex. 4. Express as a decimal of £1, (i) 7s. 7d. and (ii) 13s. 5d.

$$(i) \text{ 7s. 7d.} = 91 \text{ pence} = £\frac{91}{240} = £\frac{9.1}{24}.$$

Carrying out the division, using first
the factors 2 or 5, or multiples of them,

$$7\text{s. 7d.} = £0.379166\dots$$

$$\begin{array}{r|l} 4 & 9.100000\dots \\ 6 & 2.275000\dots \\ \hline & 0.379166\dots \end{array}$$

Now $£0.001 = £\frac{1}{1000} = \frac{960}{1000}$ or $\frac{24}{25}$ of a farthing.

Hence, £0.001 is *less* than a farthing, so that, in expressing any sum of money as a decimal of £1, it is only necessary to retain *three* places of decimals.

As 0.379 is nearer 0.379166... than 0.380 ;

$$\therefore 7\text{s. 7d.} = £0.379.$$

This result is said to be correct to three decimal places.

$$(ii) \text{ 13s. 5d.} = £\frac{161}{240} = £\frac{16.1}{24},$$

i.e.

$$\begin{array}{r|l} 4 & 16.100\dots \\ 6 & 4.02500\dots \\ \hline & 0.670833\dots \end{array}$$

Now, since 0.671 is nearer 0.670833... than 0.670 ;

$$\therefore 13\text{s. 5d.} = £0.671.$$

This result is also correct to three decimal places.

Practical methods for the decimalisation of money are considered in Section 4.2, pages 56-7.

Ex. 5. When the rate of exchange is $178\frac{3}{4}$ francs to £1, find, (i) to the nearest penny, how much English money is equivalent to 1243 francs; (ii) the equivalent of £5 10s. 6d. in francs, to the nearest tenth.

(i) Evidently the required English money

$$= \text{£} \frac{1243}{178\frac{3}{4}} = \text{£} \frac{113 \times 1243}{178 \times 4} = \text{£} \frac{452}{65}.$$

Expressing this as a decimal by division;

$$65 = 5 \times 13.$$

$$\begin{array}{r} 5 \overline{) 452.0 \dots} \\ 13 \overline{) 90.400 \dots} \\ \underline{6.953846 \dots} \end{array}$$

This decimal may be carried to as many decimal places or significant figures as desired by continuing the division, if necessary.

Thus the equivalent of the fraction $452/65$ is

6.95385 to five decimal places, or

6.954 to four significant figures.

As, however, the decimal represents pounds, three places of decimals, or four significant figures are sufficient to determine the English equivalent to the nearest penny.

Hence, the required amount of money = £6.954.

Now $\text{£}0.954 = (0.954 \times 20)\text{s.} = 19.08\text{s.},$

and $0.08\text{s.} = (0.08 \times 12)\text{d.} = 0.96\text{d.},$ which is nearly 1d

$$\therefore 1243 \text{ francs} = \text{£}6 \text{ 19s. 1d.}$$

(ii) $\text{£}5 \text{ 10s. 6d.} = \text{£}5\frac{21}{40};$

\therefore Number of francs equivalent to £5 10s. 6d.

$$= 5\frac{21}{40} \times 178\frac{3}{4} = \frac{221}{40} \times \frac{715}{4} = \frac{31603}{32} = 987.59375$$

= 987.6 to the nearest tenth of a franc.

2·7. Rules for Significant Figures.

In approximate calculation, the following important rules should always be observed :

- (a) 0 is only a significant figure when it has a digit on each side of it. In 0·05, neither 0 is significant, but in 0·0509, the 0 between the 5 and the 9 is significant.
- (b) When the first of the figures rejected is 5 or greater than 5, the last figure retained is increased by 1 ; thus 6·3812 becomes 6·4 to two significant figures, or 6·38 to three significant figures, the 8 not being increased to 9 because the next figure, 1, is less than 5.
- (c) In order to ensure that the last figure is correct, always work to at least one more figure than is asked for in the problem. As a general rule it is safer to work to two figures beyond the number required.
- (d) Always make a rough estimate of the answer first.

2·8. Long Multiplication of Decimals.

Since decimal fractions are merely an extension of the decimal notation, multiplication can be carried out in the same way as the ordinary multiplication of whole numbers. The only difference lies in placing the decimal point in the product. The rule for this can easily be deduced by first converting the decimals into ordinary fractions.

Ex. 6. *Multiply 4·53 by 3·87.*

$$\begin{aligned}\text{The product} &= 4\cdot53 \times 3\cdot87 = 4\frac{53}{100} \times 3\frac{87}{100} = \frac{453}{100} \times \frac{387}{100} \\ &= \frac{175311}{10000} = 17\cdot5311.\end{aligned}$$

From this example it is clear that the multiplication may be carried out as though there were no decimals and then the position of the decimal point in the product may be fixed by noticing that the number of figures to the right of the decimal point is equal to the sum of numbers of figures to the right of the decimal points in the two numbers to be multiplied respectively. Without the con-

version to ordinary fractions, the working may effectively be shown as follows :

4·53	2 figures to the right of the decimal point,
<u>3·87</u>	2 " " " " " "
13·59	
3·624	
·3171	∴ in the product, there are
<u>17·5311</u>	2 + 2 or 4 figures to the right of the decimal point.

∴ $4·53 \times 3·87 = 17·5311$.

2·9. Approximate Multiplication.

When the given numbers to be multiplied are only approximately true, it is necessary to determine to what extent their product is reliable.

Ex. 7. *The area of the rectangle is found by multiplying the length by the breadth. Calculate the area of a rectangle whose measured length and breadth are approximately 5·76 ft. and 4·38 ft. respectively, these measurements each being correct to three significant figures.*

From the rule given, the area in square feet is

$$5·76 \times 4·38 = 25·2288.$$

As the given measurements are each correct to three significant figures, it is necessary to determine to how many significant figures this product is reliable.

Now since 5·76 is correct to three significant figures, the number from which it has been approximated might lie anywhere between 5·764 and 5·755, so that the greatest length of the rectangle may be 5·764 feet and the least 5·755 feet.

Similarly, the greatest and least values of the breadth are 4·384 feet and 4·375 feet respectively.

Hence, the area in square feet lies between

$$\begin{array}{l} 5·764 \times 4·384 \quad \text{and} \quad 5·755 \times 4·375; \\ \text{i.e. between} \quad 25·269376 \quad \text{and} \quad 25·178125, \end{array}$$

or, taking each of these numbers correct only to three significant figures, between

$$25\cdot3 \quad \text{and} \quad 25\cdot2.$$

The area first calculated, correct to three significant figures, is, in square feet, $25\cdot2$. But the area might be $25\cdot3$ sq. ft., from the above, so that the calculated value is not even correct to three significant figures.

If, however, the areas are taken only to two significant figures, the greatest value is 25 sq. ft., the least value is 25 sq. ft. and the calculated value is 25 sq. ft. ; thus there is no variation and the calculated value is therefore only reliable when taken to *two* significant figures, although the given measurements were true to *three* significant figures.

$$\therefore \text{ the area} = 25 \text{ sq. ft.}$$

It may be shown generally that, if n denotes any whole number, the product of two approximate numbers, each given correct to n significant figures, is only reliable to $(n - 1)$ significant figures.

In the multiplication of approximate numbers, the above rule renders it possible to use only those figures which are necessary to obtain a reliable result. In Ex. 7, since the given measurements are correct to three significant figures, it will be sufficient to work only to four figures, in accordance with (c), page 25, to ensure a result correct to $3 - 1$ or 2 significant figures. The actual working may be carried out as follows :

$$\begin{array}{r} 5\cdot76 \\ 4\cdot38 \\ \hline 23\cdot04 \\ 1\cdot71 \cdot \\ 0\cdot40 \cdot\cdot \\ \hline 25\cdot15 \cdot\cdot \end{array}$$

Place the unit (4) of the multiplier under the last figure (6) of the number to be multiplied, the decimal points will then lie vertically under that of the number multiplied. After multiplying by 4, draw a vertical line immediately to the right of the fourth figure (4) of the product ; proceed with the multiplication, rejecting all figures to the right of this line, rejected figures being indicated by dots. Proceeding in this way, a four-figure product ($25\cdot15$) is obtained and, taking this correct to two

significant figures, the answer becomes 25, which agrees with that already found.

This shortened form is known as **Contracted Multiplication**.

EXERCISES 2A

1. Express (i) 16s. 7d. as a decimal of £1, (ii) 3 qr. 10 lb. as a decimal of 1 cwt., giving each result correct to four significant figures. 82.2 839

2. Express as a decimal of £1 correct to three places of decimals :

(i) 9s. 5½d., (ii) 11s. 7d. 579 (U.L.C.I.)

3. Give the exact value in £ s. d. of each of the following :

(i) £6.75625, (ii) £96.325, (iii) £8.1625. (U.L.C.I.)

4. Express each of the following amounts as a decimal of £1 correct to three places of decimals :

(i) 2s. 4½d., (ii) 7s. 9½d., (iii) 12s. 2d. (U.L.C.I.)

5. Express each of the following in cwt., qr., lb. :

(i) 0.61758 ton, (ii) 0.50967 ton.

6. Express £15 14s. 7d. as a decimal of 7 guineas, correct to six significant figures. (L.Ch.C.)

7. Express in shillings and pence :

(i) £0.7375, (ii) £0.068. (U.L.C.I.)

8. Express (i) 18s. 3d. as a decimal of £1 and (ii) 13 cwt. 2 qr. 21 lb. as a decimal of one ton, giving each result to four decimal places. Hence, find to the nearest penny, the cost of 2 tons 13 cwt. 2 qr. 21 lb. of coal at £1 18s. 3d. per ton. (U.L.C.I.)

9. Multiply 46.78 by 1.357, giving the product correct to three significant figures.

10. Work out the product of 67.327 and 59.861 correctly to five significant figures.

11. By contracted multiplication, multiply 673.24 by 4.5435 correct to six significant figures.

12. Evaluate 57.53×0.837 , giving the result correct to five significant figures.

13. Find the product of 8.3465 and 0.46235 correctly to two places of decimals.

14. Calculate the value of $5.72 \times 5.72 \times 3.14$ correctly to five significant figures.

15. Work out the value of $3.34 \times 3.34 \times 3.14$ to the nearest unit.

16. Calculate the cost of 25860 cubic feet of gas at 8.6d. per therm, taking five therms to 1000 cubic feet.

17. Find, to the nearest franc, the difference between £156 17s. 6d. and 23000 francs when £1 = 147.23 francs. (L.Ch.C.)

18. If 1 ton = 1.01605 metric tons, calculate the number of metric tons in 78.43 tons, giving the result correct to three significant figures.

19. Given that 1 metre = 1.0936 yards, express 1 kilometre in miles correct to three places of decimals, having given that 1 kilometre = 1000 metres.

20. At one time the exchange rate with Warsaw was 43.38 zloty to £1. Later, it became $25\frac{5}{8}$. An English shipping clerk in a Warsaw office received a salary of £287 per annum when the exchange was at the higher rate. What would he have to be paid per annum, to the nearest penny, to be as well off when the rate of exchange fell to the lower figure?

21. Find, to three significant figures, the number of cubic feet in 100 litres, having given that 1 litre = 0.21997 gallon and 1 gallon = 0.1604 cubic feet.

22. Express a price of 6s. 8d. per lb. in francs per kilogram, correct to four significant figures, if £1 = $178\frac{3}{4}$ francs and 1 kilogram = 2.2 lb.

23. Simplify $\frac{32}{91} + \frac{15}{77} - \frac{20}{143}$, and give the result as (i) an ordinary fraction in its lowest terms, and (ii) a decimal correct to three significant figures.

24. Express 3 qr. 4 lb. as a decimal of 1 cwt. correct to four decimal places; hence calculate, to the nearest penny, the import duty on 83 cwt. 3 qr. 4 lb. at 6s. 6.8d. per cwt.

25. Calculate, to the nearest penny, the value of

$$£\{352 \times (1.035)^2\}.$$

26. Given that a square metre = 1.196836 square yards, 1 hectare = 10000 square metres and 1 acre = 4840 square yards, calculate, to three significant figures, the number of acres equivalent to 1 hectare.

2·9. Long Division of Decimals.

The simple rule for the long division of whole numbers applies equally to numbers involving decimals. To fix the position of the decimal point in the quotient, it is necessary to convert the divisor into a whole number. The process may best be explained by an example.

Ex. 8. *Divide 5·02 by 74·8 correctly to three places of decimals.*

To obtain a rough estimate of the answer, express each number correct to the nearest whole number, then

$$5·02 \div 74·8 \quad \text{or} \quad \frac{5·02}{74·8} \quad \text{becomes} \quad \frac{5}{75} = \frac{1}{15} = 0·066 \dots$$

To carry out the actual division, the divisor must first be converted into a whole number. In this case it must be multiplied by 10; but the number to be divided must also be multiplied by 10, otherwise the quotient would be one-tenth of its actual value.

$$\text{Hence,} \quad \frac{5·02}{74·8} = \frac{5·02 \times 10}{74·8 \times 10} = \frac{50·2}{748}.$$

The actual division may then be set out as follows :

0·0671 ...	By writing the quotient above the number
748) 50·2000 ...	divided, there is no difficulty in fixing the
<u>44·88</u>	position of the decimal point, for it lies
5·320	vertically above the point in that number.
<u>5·236</u>	
0·0840	In other respects, the working is precisely
<u>0·0748</u>	the same as for the division of whole numbers.
92	

$$\therefore 5·02 \div 74·8 = 0·067 \text{ to three places.}$$

This value may be described as correct to *two* significant figures.

2·10. Approximate Division.

In actual practice, especially when dealing with large numbers, it is necessary to use methods of approximation, as in multiplication. These are considered in the following examples.

Ex. 9. Divide 9.7742 by 37.831. If the given numbers are each correct to five significant figures, determine how many significant figures in the quotient are reliable.

Roughly, writing 10 for 9.7742 and 40 for 37.831, the quotient will be $10 \div 40 = \frac{1}{4} = 0.25$, which gives an idea of the magnitude of the quotient expected.

Converting the divisor into a whole number,

$$\frac{9.7742}{37.831} = \frac{9.7742 \times 1000}{37.831 \times 1000} = \frac{9774.2}{37831}.$$

Hence, by the usual process, the working is as follows :

$$\begin{array}{r} 0.258364 \dots \\ 37831 \overline{) 9774.200000 \dots} \\ \underline{7566.2} \\ 2208.00 \\ \underline{1891.55} \\ 316.450 \\ \underline{302.648} \\ 13.8020 \\ \underline{11.3493} \\ 2.45270 \\ \underline{2.26986} \\ .182840 \\ \underline{.151324} \\ 31516 \end{array}$$

To determine how many figures of this quotient are reliable, it must be observed that, since 9.7742 is correct to five significant figures, it must lie between 9.77424 and 9.77415.

Similarly, 37.831 must lie between 37.8314 and 37.8305.

Hence, the quotient must lie between

$$\frac{9.77424}{37.8305} \quad \text{and} \quad \frac{9.77415}{37.8314}$$

taking the extreme cases.

Carrying out each division to six places, the actual quotient must lie between

$$0.258369 \quad \text{and} \quad 0.258361.$$

Comparing these with the quotient already found,

$$0.258369 > 0.258364 > 0.258361,$$

and to five significant figures,

$$0.25837 > 0.25836 = 0.25836.$$

Finally, to four significant figures,

$$0.2584 = 0.2584 = 0.2584,$$

so that the quotients now show no variation.

Hence, $9.7742 \div 37.831 = 0.2584.$

Thus, although the given numbers are correct to *five* significant figures, the quotient is correct only to *four* significant figures.

Ex. 10. *The total quantity of coal exported in 1927 was 248,870,356 tons and its value is given as £169,764,458 ; calculate the value per ton as a decimal of £1 correct to three significant figures.*

The required value is $\text{£}169,764,458 \div 248,870,356.$

Roughly, this is $\text{£}\frac{17}{25} = \text{£}\frac{68}{100} = \text{£}0.68.$

Working the long division in full, thus :

$$\begin{array}{r}
 0.6821401 \dots \\
 248870356 \) \ 169764458.0000000 \dots \\
 \underline{149322213.6} \\
 20442244.40 \\
 \underline{19909628.48} \\
 532615.920 \\
 \underline{497740.712} \\
 34875.2080 \\
 \underline{24887.0356} \\
 9988.17240 \\
 \underline{9954.81424} \\
 33.3581600 \\
 \underline{24.8870356} \\
 8.4711244
 \end{array}$$

Since, however, the answer is to be correct to three significant figures, the work may be considerably shortened by using only *four* significant figures of the given numbers : thus, correct to four significant figures,

$$\begin{array}{l}
 248870356 = 248900000 \\
 \text{and } 169764458 = 169800000
 \end{array}$$

But

$$\frac{169800000}{248900000} = \frac{1698}{2489},$$

so that the shortened form of the division may be used. This is shown at the top of page 33 where it will be seen that the quotient 0.68220... agrees with the previous quotient 0.68214... to three significant figures only.

Hence, the required value = **£0.682.**

0.68220... Note that in the division of decimals :

2489) 1698.00000...

1493.4

204.60

199.12

5.480

4.978

5020

4978

420

(i) The divisor must always be made a whole number.

(ii) The decimal point in the dividend must be moved as many places to the right as there were decimal figures in the divisor before it was made a whole number.

(iii) If an answer is required to n significant figures, it is only necessary to work with $(n + 1)$ significant figures.

2.11. Combination of Multiplication and Division.

Problems are often met with in practice which give rise to combined multiplication and division of decimals.

Ex. 11. Evaluate

$$\frac{0.81742 \times 7.483}{24.864}$$

correctly to three significant figures.

$$\text{Roughly, the fraction} = \frac{0.8 \times 7.5}{25} = \frac{6}{25} = 0.24.$$

Since the result is required to be correct to three significant figures, the numbers in the final division must be correct to four figures at least ; as the divisor, 24.864, is presumably correct to five figures, the product of 0.81742 and 7.483 must be worked to five figures ; that is, the multiplication must be taken to six figures at least. Hence the following working :

0.81742

7.483

5.72194

.32696

6536

243

6.11669

.

..

...

...

Now

$$\frac{6.1167}{24.864} = \frac{6116.7}{24864}$$

= 6.1167 to five figures.

$$\begin{array}{r}
 0\cdot24600\dots \\
 24864 \) \ 6116\cdot70000\dots \\
 \underline{4972\cdot8} \\
 1143\cdot90 \\
 \underline{994\cdot56} \\
 149\cdot340 \\
 \underline{149\cdot184} \\
 \cdot15600
 \end{array}$$

Hence the required result = $0\cdot246$.

EXERCISES 2B

1. Divide $9\cdot8121$ by $37\cdot831$, giving the quotient correct to four significant figures.

2. 1,488,173 persons in England and Wales received poor relief during 1933, the total expenditure being £38,923,852. What was the average amount received by each person, correct to the nearest penny? (L.Ch.C.)

3. In 1934, the gold production in South Africa was 10,479,857 fine ounces to a total value of £72,311,013. Find the average price per fine ounce correct to the nearest penny. (L.Ch.C.)

4. In a certain year, the Customs duty paid on 97,856,416 lb. of tobacco was £58,102,247 ; calculate the duty levied per lb.

5. A branch of a Co-operative Society has 3616 members whose purchases for a half-year totalled £58,750. Find the average sales per member, to the nearest penny. (U.L.C.I.)

6. In 1919, 229,779,517 tons of coal were raised by 1,191,313 miners. Find the average number of tons raised per miner correct to four significant figures.

7. From the following table, which relates to the county of Essex, calculate

- (i) the increase in population in 1931, correct to two decimal places, per 100 of the population in 1928 ;
- (ii) the density of the population, that is, the number of inhabitants per acre, for 1928 and 1931, in each case to two significant figures.

Year	Area in acres	Population
1928	979,532	1,478,506
1931	979,532	1,755,459

8. Evaluate $\frac{0.0716 \times 5.865}{0.6241}$ to three significant figures.
9. Using contracted multiplication and division find, correct to three decimal places, the value of $\frac{81.754 \times 0.69384}{7.3586}$. (U.L.C.I.)
10. Calculate, correctly to two places of decimals, the value of $\frac{2.17732 \times 9.08813}{1.82641}$.
11. Evaluate $\frac{10.81 \times 1.431}{0.188 \times 12.19}$.
12. Find, to the nearest halfpenny, the value of $13s. 1\frac{1}{2}d. \times 1.176 \div 1.146$.
13. Calculate the value of $\frac{0.474 \times 3.1416}{9.2}$ to three significant figures.
14. The rateable value of a district is £2,783,964 and the estimated expenditure for the half-year is £974,398. Calculate in £, to two significant figures, the rate in the £ which must be levied to raise this amount. Express the rate also in shillings and pence.
15. In a borough a sum of £94,073 was raised by a rate of 8s. 9d. in the £. Find the rate necessary to raise the same amount when the rateable value of the borough has increased by £10,000. (L.Ch.C.)
16. In a borough with a rateable value of £157,428 it was necessary to raise £75,434 from the rates. The following year the necessary amount had increased by £2500 but it was only necessary to raise the rate by 2d. in the £. What was the new rateable value? (L.Ch.C.)
17. When the value of a franc was 9.513 pence and the value of a dollar was 4s. 1.32d, calculate the number of dollars whose value was equal to that of 278.7 francs.

18. In 1937, 32,442 posts were filled in the Civil Service and during the same year, of 69,690 candidates examined for entrance, 15,858 men and 12,970 women failed. Calculate per 100 candidates

- (i) the number of men who failed,
- (ii) the number of women who failed,
- (iii) the number of posts not filled from the successful candidates.

Give each result correct to two places of decimals.

19. Lead at 4884 francs per tonne in France is sold to a dealer in London. Find the price he pays per ton, to the nearest penny, if $\text{£}1 = 178.8$ francs and $1 \text{ ton} = 1.016$ tonne.

20. The value of the Imports for November 1935 was $\text{£}71,455,232$, whilst for November 1936 it was $\text{£}78,671,360$. Calculate in £ , to three significant figures, the increase in 1936, per $\text{£}100$ of the value in 1935.

21. A district with a rateable value of $\text{£}13,076$ and rates at 13s. 3d. in the £ is absorbed by a borough with a rateable value of $\text{£}22,883$ and rates at 11s. $10\frac{1}{2}$ d. in the £ . If the same total sum is to be raised during the first half-year after the union, calculate the rate in the £ which must be raised.

22. Express a price of 23.84 francs per kilogram in shillings and pence, to the nearest penny, per lb., taking $1 \text{ kilogram} = 2.205 \text{ lb.}$ and $\text{£}1 = 178.75$ francs.

23. The production of wheat in a particular district of France in a certain year was 19.71 quintals per hectare. Express this in cwt. per acre, to three significant figures, taking $1 \text{ quintal} = 100 \text{ kilograms}$, $1 \text{ kilogram} = 2.2046 \text{ lb.}$ and $1 \text{ hectare} = 2.4711 \text{ acres.}$

CHAPTER III

BRITISH WEIGHTS AND MEASURES—THE METRIC SYSTEM

3.1. Fundamental Units.

The British Weights and Measures are based upon the **Imperial Standard Yard** and the **Imperial Standard Pound** which are defined legally by the Weights and Measures Act of 1878.

The Imperial Yard is the distance, measured at a temperature of 62° F., between two fine transverse lines engraved on two gold plugs, one on each, sunk into the ends of a bronze bar.

The Imperial Standard Pound is defined as the weight, measured in vacuum, of a certain platinum cylinder.

These standards are kept at the Government Standards Office and official copies are placed in the Houses of Parliament, the Royal Mint, the Royal Observatory at Greenwich, and at the office of the Royal Society.

3.2. The Gallon.

The unit of capacity is the **gallon** which is defined as the space occupied by 10 lb. of distilled water at 62° F. and at an air pressure of 30 inches of mercury.

In various branches of trade, particular units are in use, but they all depend fundamentally upon the Standard Yard and the Standard Pound and sometimes upon the gallon. For example,

	a <i>bag</i> of flour is 140 lb.,
whilst	a <i>bag</i> of hops is 280 lb.
Similarly,	a <i>barrel</i> of butter is 224 lb.,
whilst	a <i>barrel</i> of tar is $26\frac{1}{4}$ gallons.

Thus, various trades do use unofficial units for convenience, but, by law, they must all be expressible in terms of the yard, pound and gallon.

3.3. Imperial Measures of Length.

Linear measurements depend generally upon the following table :

(i) Length.

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards	= 1 pole (po.)
4 poles or 22 yards	= 1 chain (ch.)
10 chains or 40 poles	= 1 furlong
8 furlongs	= 1 mile (mi.)

Formerly 3 miles constituted 1 *league*, but this is now obsolete. It is evident from the above table that,

$$1 \text{ mile} = 80 \text{ chains} = 1760 \text{ yards} = 5280 \text{ feet.}$$

In land measurement, a chain 66 feet or 22 yards long containing 100 links of equal length is used. This was first introduced by an English mathematician, Edmund Gunter (1581-1626), and is often called **Gunter's chain**. Hence,

$$100 \text{ links} = 1 \text{ chain} = 22 \text{ yards.}$$

For measurements in and on the sea, the following table is in use :

(ii) Nautical Measure.

6 feet	= 1 fathom
100 fathoms	= 1 cable-length
6080 feet	= 1 nautical mile

1 *knot* is a speed of 1 nautical mile per hour.

3.4. Imperial Measure of Weight.

(i) Avoirdupois Weight.

16 drams (Av.)	= 1 ounce (oz.)
16 ounces	= 1 pound (lb.)
14 pounds	= 1 stone (st.)
2 stones or 28 lb.	= 1 quarter (qr.)
4 quarters or 112 lb.	= 1 hundredweight (cwt.)
20 cwt. or 2240 lb.	= 1 ton

For weighing meat, a *Smithfield stone* has been used ; this was 8 lb.
By law, Avoirdupois weight must be applied to all goods sold by weight, except :

(a) Precious metals and stones, to which a special weight, known as *Troy weight*, is applied.

(b) Drugs, for which *Apothecaries' weight* is used.

The tables of these are as follows :

(ii) **Troy Weight.**

$$\begin{aligned} 24 \text{ grains (gr.)} &= 1 \text{ pennyweight (dwt.)} \\ 20 \text{ dwt.} &= 1 \text{ ounce (oz. Tr.)} \end{aligned}$$

(iii) **Apothecaries' Weight.**

$$\begin{aligned} 20 \text{ grains (gr.)} &= 1 \text{ scruple} \\ 3 \text{ scruples} &= 1 \text{ drachm} \\ 8 \text{ drachms} &= 1 \text{ ounce} \end{aligned}$$

The grain was originally supposed to be the weight of a dried grain of wheat. In 1 lb. Avoirdupois there are 7000 grains.

The fineness of pure gold is taken as 24 carat, so that 18-carat gold indicates an alloy of which $\frac{18}{24}$ or $\frac{3}{4}$ is pure gold.

For gold and silver the ounce is now generally divided decimally.

Ex. 3. (i) *Shew that 1 oz. Troy = 1·097... oz. Avoirdupois.*

(ii) *What is the value of the gold in a 15-carat article weighing 16 dwt. when gold is £7 3s. 8d. per oz. Troy ?*

(i) 1 oz. Troy = 20 dwt. = 20×24 grains, and

1 oz. Avoirdupois = $7000 \div 16$ grains ;

$$\therefore \frac{1 \text{ oz. Troy}}{1 \text{ oz. Av.}} = \frac{20 \times 24 \times 16}{7000} = \frac{768}{700} = \frac{7 \cdot 68}{7} = 1 \cdot 097 \dots$$

Hence, 1 oz. Troy = 1·097... oz. Avoirdupois.

(ii) The gold in the article will weigh

$$\frac{16}{20} \times \frac{15}{24} \text{ oz. Tr.} = \frac{1}{2} \text{ oz. Tr.}$$

\therefore the value of the gold = $\frac{1}{2}$ of £7 3s. 8d. = £3 11s. 10d.

Note that 1 ounce Troy = 20×24 grains = 480 grains and the Apothecary ounce = $8 \times 3 \times 20$ grains = 480 grains,

$$\therefore 1 \text{ oz. Troy} = 1 \text{ oz. Ap.} = 480 \text{ grains.}$$

3·5. Measures of Capacity.

The Imperial Measure of Capacity for common liquids and dry goods is as follows :

(i) Imperial Measure of Capacity.

4 gills	= 1 pint (pt.)
2 pints	= 1 quart (qt.)
4 quarts	= 1 gallon (gal.)
2 gallons	= 1 peck (pk.)
4 pecks	= 1 bushel (bush.)
8 bushels	= 1 quarter (qr.)

Gills, pints, quarts and gallons are used for liquids ; pecks, bushels and quarters for dry goods. For liquids also,

$$1 \text{ gallon} = 277 \cdot 274 \text{ cubic inches}$$

and

$$1 \text{ barrel} = 36 \text{ gallons.}$$

Many special units are used in various trades ; for instance,

1 sack of flour	= 5 bushels
1 bag of flour	= 3 bushels
12 bags of flour	= 1 chaldron

$$\therefore 1 \text{ chaldron of flour} = 36 \text{ bushels} = 4\frac{1}{2} \text{ quarters.}$$

But for coal, 1 chaldron = 85 bushels, where

$$1 \text{ bushel weighs } 80 \text{ lb.}$$

For dispensing drugs, the legal measure is :

(ii) Apothecaries' Measure of Capacity.

60 minims (min.)	= 1 fluid drachm
8 fluid drachms	= 1 fluid ounce
20 fluid ounces	= 1 pint
8 pints	= 1 gallon

The weight of 1 fluid drachm is 2 drams Av.

The approximate equivalents of a tea-spoonful and a table-spoonful are 1 and 4 fluid drachms respectively.

Ex. 2. Taking the population of New York as 6,103,384, calculate (i) the number of barrels of flour needed per week, if each person consumes on an average 2 lb. 10 oz. weekly, and 1 barrel of flour weighs 196 lb., (ii) the weight of the flour in tons, cwt., lb., and (iii) the British equivalent of this weight, in bushels, taking 1 peck of flour to weigh 14 lb.

$$(i) \ 2 \text{ lb. } 10 \text{ oz.} = 2\frac{5}{8} \text{ lb.}$$

$$\therefore \text{ number of lb. consumed} = 6,103,384 \times 2\frac{5}{8}$$

$$\text{and the number of barrels needed} = 6,103,384 \times 2\frac{5}{8} \div 196$$

$$\begin{aligned} &= \frac{6103384 \times 21}{196 \times 8} = \frac{108989 \times 3}{4}, \text{ on cancelling} \\ &= \frac{326967}{4} = 81741\frac{3}{4}. \end{aligned}$$

$$\therefore \text{ number of whole barrels actually needed} = 81742.$$

$$\begin{aligned} (ii) \text{ The exact weight} &= \frac{6103384 \times 21}{8} \text{ lb.} = \frac{6103384 \times 21}{112 \times 8} \text{ cwt.} \\ &= \frac{762923 \times 3}{16} \text{ cwt.} = \frac{2288769}{16} \text{ cwt.} \\ &= 143048\frac{1}{16} \text{ cwt.} \\ &= 7152 \text{ tons } 8 \text{ cwt. } 7 \text{ lb.} \end{aligned}$$

(iii) Since 1 peck of flour weighs 14 lb. ;

$$\therefore 1 \text{ bushel will weigh } (14 \times 4) \text{ lb.} = 56 \text{ lb.} = \frac{1}{2} \text{ cwt.}$$

Hence, the number of bushels equivalent to the weight just found in (ii) = $143048\frac{1}{16} \times 2 = 286096\frac{1}{8}$.

3·6. The Metric System.

Many exercises have already been given involving some of the more commonly used units of the Metric System. It will now be necessary to consider the system more fully.

The Metric System of measurement was adopted by France in 1801, and is a decimal system. It is in general use in most of the

countries of Europe and is universally used in scientific work. The units of length, weight and capacity are called respectively the **Metre**, the **Gram** and the **Litre**. The metre was originally intended to be one ten-millionth part of the distance from the North Pole to the Equator, measured along the meridian through Paris. It was found, however, that the distance was very difficult to measure accurately, and so a platinum bar was constructed as near this length as possible, and is preserved as a permanent standard of the length.

The metric table of linear measure is as follows :

10 millimetres (mm.)	= 1 centimetre (cm.)
10 centimetres	= 1 decimetre (dm.)
10 decimetres	= 1 metre (m.)
10 metres	= 1 Dekametre (Dm.)
10 Dekametres	= 1 Hectometre (Hm.)
10 Hectometres	= 1 kilometre (km.)

Note that *kilo-* = 1000, *Hecto-* = 100, *Deka-* = 10, whilst *deci-* = $\frac{1}{10}$ = 0·1, *centi-* = $\frac{1}{100}$ = 0·01, and *milli-* = $\frac{1}{1000}$ = 0·001. The terms *deci-*, *Hecto-* and *Deka-* are seldom used, and lengths are generally expressed in terms of one unit only ; thus 14 Dekametres is usually written as 140 metres or 0·140 kilometre.

Ex. 3. Express each of the following lengths in metres and find their sum correct to four significant figures :

22063 mm.,	109·7 dm.,	0·1348 km.,	45·7 cm.
22063 mm.	=(22063÷1000) m. = 22·063 m.		
109·7 dm.	=(109·7÷10) m. = 10·97 m.		
0·1348 km.	=(0·1348×1000) m.=134·8 m.		
45·7 cm,	=(45·7÷1000) m. = <u>0·457 m.</u>		
			168·290 m.

∴ the sum, correct to four significant figures = **168·3 metres**.

The Gram (gm.) is the weight of one cubic centimetre, i.e. a cube of one centimetre edge (Fig. 2), of distilled water, whilst a Litre (l.)

is the capacity of 1000 cubic centimetres. If in the above table of linear measure, *grams* be written for metres, or *litres* for metres, the corresponding tables of weight and capacity respectively are given; and again the terms deci-, Hecto-, and Dekka- are rarely used. Thus weights are expressed as *kilograms* and *grams* whilst very large weights are expressed in *tonnes*, 1 tonne being equivalent to 1000 kilograms. In weighing grain, a unit known as a *quintal* is used. 1 quintal = $\frac{1}{10}$ tonne = 100 kilograms, and 1 kilogram = 2.2046 lb. correct to four decimal places.



FIG. 2.—1 cubic centimetre.

1 lb. is approximately equal to 453.6 gm., and 1 litre is very nearly equal to $1\frac{3}{4}$ pints. For liquid and dry measure the litre only is used.

Ex. 4. *A chest of tea, weighing 42.85 kgm., is sent to London where the tea is made up in 1 lb. packets. If the chest when empty weighs 4747.6 grams and 1 lb. is equivalent to 453.6 grams, find the number of packets made up.*

$$\text{Weight of (chest + tea)} = 42.85 \text{ kgm.}$$

$$\text{Weight of empty chest} = 4747.6 \text{ gm.} = 4.7476 \text{ kgm.}$$

$$\therefore \text{weight of tea} = 38.1024 \text{ kgm.} \\ = 38102.4 \text{ grams.}$$

Since 1 lb. = 453.6 grams,

$$\therefore \text{number of 1 lb. packets of tea made up} = \frac{38102.4}{453.6} = \frac{381024}{4536} \\ = 84.$$

It may be noted that a chest of tea usually contains 84 lb.

From Exercises 3 and 4 it will be seen that, in addition and subtraction of decimals, the numbers are written down exactly as in ordinary addition and subtraction, i.e. the units are written under one another in one column, then the decimal points are also vertically under one another.

3·7. Imperial and Metric Equivalents.

It is frequently necessary in practice to convert one set of units into another and, for this purpose, appropriate tables of conversion factors are needed. The following tables are therefore provided to give the more important equivalents between fundamental units most commonly used of the Imperial and the Metric Systems. Other equivalents will be given later as required.

Table of Equivalents

(i) <i>British to Metric</i>	(ii) <i>Metric to British</i>
1 inch = 2·5400 cm.	1 centimetre = 0·3937 in.
1 yard = 0·9144 m.	1 metre = 1·0936 yd.
1 mile = 1·6093 km.	1 kilometre = 0·6214 mi.
1 oz. Av. = 28·3495 gm.	1 gram = 15·4323 gr.
1 lb. = 453·5924 gm.	1 kilogram = 2·2046 lb.
1 ton = 1·0160 metric tons	1 metric ton = 0·9842 ton
1 gallon = 4·5460 litres	1 litre = 1·598 pt.

Ex. 5. Using the above table of equivalents, calculate (i) the number of grams equivalent to 1 oz. Troy, (ii) the number of litres equivalent to a bushel.

(i) Since 1 lb. Av. = 7000 grains,

$$\therefore 1 \text{ oz. Av.} = \frac{7000}{16} \text{ grains.} \quad \text{There is no need to work this}$$

out as the following steps will shew.

$$\text{But} \qquad 1 \text{ oz. Tr.} = 480 \text{ grains,}$$

$$\text{and} \qquad 1 \text{ oz. Av.} = 28·3495 \text{ grams,}$$

$$\text{i.e.} \qquad \frac{7000}{16} \text{ grains} = 28·3495 \text{ grams.}$$

$$\text{Hence,} \quad 1 \text{ oz. Tr., or 480 grains} = \frac{28·3495 \times 480 \times 16}{7000} \text{ grams}$$

$$= \frac{0.283495 \times 768}{7} \text{ grams} = \frac{217.72416}{7} \text{ grams}$$

$$= 31.10345 \dots \text{ grams.}$$

∴ to four places of decimals,

$$1 \text{ oz. Troy} = 31.1035 \text{ grams.}$$

$$(ii) \quad 1 \text{ bushel} = 8 \text{ gallons (p. 40)}$$

$$= 8 \times 4.5460 \text{ litres (p. 44)}$$

$$= 36.368 \text{ litres.}$$

$$\text{Otherwise : } 1 \text{ bushel} = 64 \text{ pints} = \frac{64}{1.7598} \text{ litres (p. 44)}$$

$$= \frac{640000}{17598} \text{ litres} = 36.368 \text{ litres.}$$

Hence, to three places of decimals,

$$1 \text{ bushel} = 36.368 \text{ litres.}$$

3·8. British Money.

Money is a measure of value as well as a medium of exchange, and gold serves as a convenient standard by which many basic coins may be compared in actual or intrinsic value. British standard gold is measured in carats, 24 carats representing pure or fine gold. The Currency Law of 1816 fixed the gold standard of Great Britain as the **Pound Sterling**, and, in order to render this coin durable, the gold was alloyed with a baser metal such that, in 12 parts by weight, 11 were fine gold and 1 alloy. The gold is thus said to be $\frac{11}{12}$ fine, so that a sovereign is thus 22 carats fine. This fineness is taken as **standard gold**.

The weight of a sovereign is 123.274 grains or 7.98805 grams ; therefore, the weight of fine gold in a sovereign is

$$\frac{11}{12} \text{ of } 123.274 \text{ grains} \quad \text{or} \quad \frac{11}{12} \text{ of } 7.98805 \text{ grams}$$

i.e. $\quad 113.001 \text{ grains} \quad \text{or} \quad 7.32238 \text{ grams.}$

British silver coins also do not consist wholly of pure silver.

Formerly, in 40 parts by weight, 37 were silver and 3 alloy; the silver was thus said to be $\frac{37}{40}$ fine. Since 1920, however, the fineness has been reduced to $\frac{1}{2}$, so that silver coins now contain only one part in two of pure silver.

The approximate weights of the more commonly used British silver coins are as follows :

Half-crown	-	-	-	-	218.18181 grains.
Florin or Two-shilling piece	-	-	-	-	174.54545 „
Shilling	-	-	-	-	87.27272 „
Sixpence	-	-	-	-	43.63636 „

The bronze coins consist of an alloy of 95 parts of copper, 4 parts of tin and 1 part of zinc. The weights of the coins are :

Penny	-	-	-	-	145.833 grains = $\frac{1}{3}$ oz. Av.
Half-penny	-	-	-	-	87.500 „ = $\frac{1}{6}$ „
Farthing	-	-	-	-	43.750 „ = $\frac{1}{10}$ „

Incidentally, the diameter of a half-penny is exactly one inch.

It may be observed that since 1914 sovereigns and half-sovereigns have been replaced by Treasury Notes representing the respective values of those coins.

Ex. 6. *In making 1869 sovereigns, 480 ounces Troy of standard gold were formerly used. Calculate the value of fine gold per ounce Troy.*

Since standard gold is $\frac{11}{12}$ fine,

\therefore weight of fine gold is 480 oz. Troy = $\frac{11}{12}$ of 480 oz. = 440 oz.

\therefore value of 1 oz. of fine gold = $\pounds \frac{1869}{440}$ = $\pounds 4.24772 \dots$

= $\pounds 4$ 4s. $11\frac{1}{2}$ d.

3.9. Foreign Money.

Many foreign countries use a decimal system of coinage based upon the French Franc. This was formerly a silver coin containing 83.5 parts of silver in 100 and weighing 5 grams; its actual value in British money was 9.513 pence or about $9\frac{1}{2}$ d. Since 1928, however, the coin contains 0.0655 of gold 0.9 fine.

Originally the franc was divided into *décimes*, *sous* and *centimes*; the relations between the values of these coins were:

$$\begin{aligned} 1 \text{ sou} &= 5 \text{ centimes} \\ 10 \text{ centimes} &= 1 \text{ décime} \\ 10 \text{ décimes} &= 1 \text{ franc} \end{aligned}$$

When the franc was equivalent in value to $9\frac{1}{2}$ d., the value of the sou was $(9\frac{1}{2} \div 20)$ pence or very nearly one half-penny. Now it is very much less, as the value of the franc has considerably diminished. Ex. 7 will show this.

The *décime* is not now in use.

Ex. 7. *By comparing the weights of gold in a sovereign and a franc, find the precise value of (i) £1 in francs, (ii) a franc in pence.*

(i) In Section 3·8, it was stated that the sovereign contains 7·32238 grams of fine gold, and from Section 3·9, a franc contains 0·0655 gram of gold 0·9 fine,

$$\text{i.e.} \quad 0\cdot0655 \times 0\cdot9 \text{ gram of fine gold ;}$$

\therefore the number of francs equivalent in gold value to £1

$$\begin{aligned} &= \frac{7\cdot32238}{0\cdot0655 \times 0\cdot9} = \frac{7\cdot32238}{0\cdot05895} = \frac{732238}{5895} \\ &= 124\cdot21. \end{aligned}$$

Hence, **124·21 francs are equal in value to £1.**

This is known as the *intrinsic exchange value*.

(ii) From the above result, it is evident that the value of

$$1 \text{ franc in pence} = \frac{240}{124\cdot21} = 1\cdot932.$$

\therefore **intrinsic value of 1 franc = 1·932 pence.**

It will thus be seen that the intrinsic value of the new franc is considerably less than that of the former franc.

The intrinsic rate of exchange between the Pound Sterling and a basic coin of a foreign country, determined by a comparison of the weights of fine gold contained in each, where possible, is known as the **Mint Par of Exchange**.

Thus from (i) above, the Mint Par of Exchange between the £ and the franc is the number of francs, viz. 124·21, exactly equivalent in gold value to £1.

The law of supply and demand, however, causes the exchange values between the various basic coins to fluctuate considerably from the nominal parity calculated as the Mint Par of Exchange, as will be seen later.

3-10. Some Basic Coins of Foreign Countries.

The system of dividing the basic coin into 100 parts of equal value is in use in many countries, a few of which are given below.

Coins of Foreign Countries.*

Country	Basic coin	No. to £1 M.P.E.	Sub-divisions
Belgium	Belga	35	1 Belga = 5 Francs
Denmark	Krone	18·159	1 Krone = 100 Ore
France	Franc	124·21	1 Franc = 100 Centimes
Germany	Reichsmark	20·43	1 Reichsmark = 100 Reichspfennige
Greece	Drachma	375	1 Drachma = 100 Lepta
Italy	Lira	92·46	1 Lira = 100 Centesimi
Poland	Zloty	43·38	1 Zloty = 100 Grosz
Portugal	Escudo	110	1 Escudo = 100 Centavos
Spain	Peseta	25·225	1 Peseta = 100 Centesimos
Sweden	Krona	18·159	1 Krona = 100 Ore
Switzerland	Franc	25·2215	1 Franc = 100 Centimes
U.S.A.	Dollar	4·866	1 Dollar = 100 Cents
Yugoslavia	Dinar	276·316	1 Dinar = 100 Paras

* June 1939. M.P.E. = Mint Par of Exchange.

Ex. 8. *An English merchant in New York has to pay two bills, one in Brussels for 5075 belgas and the other in Oslo for 5208 kroner. He therefore cashes a cheque for as many dollars as will just meet these debts, the respective rates of exchange being, at the time, 4·88 dollars, 29 belgas and 19·84 kroner to £1. Find the number of dollars required.*

Here it will be necessary first to convert the debts into English money and then determine the number of dollars equivalent to their sum.

Now, $5075 \text{ belgas} = £(5075 \div 29) = £175,$
and $5208 \text{ kroner} = £(5208 \div 19\cdot84) = £262\cdot5$

\therefore total value of the two debts in English money

$$= £(175 + 262\cdot5) = £437\cdot5.$$

Hence, the number of dollars equivalent to this amount

$$= 437\cdot5 \times 4\cdot88 = 2135.$$

EXERCISES 3

Section I. Mental—Tots.

In working the following exercises, only the answers required should be written down on paper. Add as quickly as possible in order to gain facility in reckoning up long columns of figures both correctly and speedily.

Find the totals of each of the following :

1.			2.			3.			
£	s.	d.	yd.	ft.	in.	tons	cwt.	qr.	lb.
463	18	11	213	2	10	11	17	2	23
27	15	7	186	1	3	27	16	1	13
216	13	10	27	1	4	3	8	3	19
59	6	3	516	1	7	0	19	1	4
127	12	7	86	1	11	44	13	2	25
583	17	8	138	2	1	2	14	3	22
87	18	1	492	1	3	0	12	1	8
276	13	5	97	2	9	8	15	2	17

In each of the Exercises 4-10, find

- (i) the horizontal totals, (a), (b), (c), etc.,
 (ii) the vertical totals, *A*, *B*, *C*,
 (iii) the grand total, G.T.

4.	£	s.	d.	£	s.	d.	£	s.	d.	
	215	14	3	54	18	11	123	19	8(a)
	5	6	7	79	3	8	45	13	0(b)
	83	11	9	116	9	2	97	8	5(c)
	176	19	4	7	12	5	243	0	9(d)
	43	8	11	238	17	6	18	14	10(e)
	29	16	5	31	2	3	3	15	7(f)
	137	2	1	66	8	10	156	5	2(g)
	<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(U.L.C.I.)

5.	£	s.	d.	£	s.	d.	£	s.	d.	
	215	14	9	148	9	5	112	13	3(a)
	71	11	8	283	7	7	98	12	8(b)
	49	3	5	3	18	1	279	5	7(c)
	163	7	10	79	15	11	56	4	9(d)
	28	15	2	343	5	4	8	11	5(e)
	5	8	3	274	8	9	87	2	4(f)
	139	14	11	63	4	8	315	9	10(g)
	<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(U.L.C.I.)

6.	£	s.	d.	£	s.	d.	£	s.	d.	
	167	11	3	209	14	7	126	13	11(a)
	29	5	8	9	3	4	183	18	3(b)
	211	9	6	57	15	8	79	12	9(c)
	6	13	5	162	17	2	43	3	4(d)
	83	16	9	319	1	9	4	8	7(e)
	345	8	10	63	19	1	256	15	2(f)
	53	18	2	38	4	10	115	9	6(g)
	<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(U.L.C.I.)

£	s.	d.	£	s.	d.	£	s.	d.	
7. 783	17	4	237	9	7	1473	11	8(a)
1239	8	9	562	12	9	649	6	9(b)
637	14	10	813	13	5	783	13	2(c)
956	5	3	1276	18	3	235	15	7(d)
2143	19	8	413	7	6	566	14	6(e)
567	12	5	927	3	11	741	7	4(f)
814	4	7	1472	14	8	384	6	5(g)
1398	17	6	359	19	7	892	11	3(h)
<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(R.S.A.)

£	s.	d.	£	s.	d.	£	s.	d.	
8. 4192	11	7	325	4	7	117	4	10 $\frac{1}{2}$(a)
853	13	5	4117	15	4	88	15	4 $\frac{3}{4}$(b)
217	8	10	7234	6	8	529	11	7(c)
1736	17	3	159	17	9	73	16	8 $\frac{1}{4}$(d)
2457	16	5	492	12	6	729	8	6 $\frac{1}{2}$(e)
920	3	2	1586	19	10	472	13	7 $\frac{3}{4}$(f)
1189	15	8	814	13	9	59	7	10(g)
3726	12	9	2354	8	11	239	16	1 $\frac{1}{2}$(h)
<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(R.S.A.)

£	s.	d.	£	s.	d.	£	s.	d.	
9. 1462	18	4	2586	4	10	3016	14	3(a)
237	16	11	37	8	8	18	13	11(b)
9	15	2	421	9	7	932	7	5(c)
4873	3	10	896	18	6	1673	8	6(d)
456	7	9	1473	14	11	1142	19	10(e)
1120	17	10	5061	7	5	10	12	9(f)
13	12	11	86	15	9	763	5	3(g)
876	19	4	549	19	7	854	1	4(h)
537	11	5	9	3	11	1467	11	10(i)
2508	8	7	1683	1	8	73	15	11(k)
3624	16	6	847	18	1	415	17	7(l)
738	13	11	13	12	6	2569	6	9(m)
<i>A</i>			<i>B</i>			<i>C</i>		G.T.

£	s.	d.	£	s.	d.	£	s.	d.	
10. 7215	16	3	6313	12	2	1302	6	7(a)
3572	9	7	4571	16	8	513	5	10(b)
5899	7	8	643	9	8	851	17	9(c)
923	13	5	2081	7	9	896	17	10(d)
384	12	4	527	16	11	1139	8	9(e)
8171	4	7	7049	4	6	472	15	4(f)
967	15	6	128	17	10	383	10	9(g)
2867	17	9	5673	14	5	726	19	5(h)
9358	9	11	3328	19	7	389	17	11(i)
7805	19	6	9614	5	8	564	8	9(k)
18	12	8	2597	13	1	94	5	7(l)
2793	5	4	3261	14	10	523	12	8(m)
<i>A</i>			<i>B</i>			<i>C</i>		G.T.

(R.S.A.)

Section II. Written.

11. Find the total cost of the following :

5 cwt. of broken coke at 39s. per ton.

6 cwt. of coal at 45s. per ton.

 $1\frac{1}{2}$ cwt. of anthracite stove nuts at 70s. per ton. (R.S.A.)

12. Find the total cost of the following purchases :

9 lb. of raisins at $8\frac{3}{4}$ d. per lb.7 lb. of sugar at $4\frac{1}{4}$ d. per lb. $\frac{1}{4}$ lb. of cocoa at 2s. 2d. per lb. $1\frac{1}{4}$ lb. of cheese at 1s. 1d. per lb. (R.S.A.)

13. Make out the following account and deduct 6d. in the £ for cash payment :

1 cwt. of tea at 2s. $3\frac{1}{2}$ d. per lb. $12\frac{1}{2}$ lb. of coffee at 2s. 8d. per lb. $1\frac{1}{2}$ cwt. of flour at 1s. 5d. per stone. $\frac{1}{2}$ cwt. of rice at $2\frac{3}{4}$ d. per lb.

9 lb. of butter at 1s. 6d. per lb.

14. Make out an invoice for the following books :

355 copies at 4s. 6d. each, less 3s. in the £.

133 copies at 3s. 9d. each, less 2d. in the shilling.

53 copies at 7s. 6d. each net.

15. The following statement, relating to the prices and cost of apples, is incomplete ; give the correct results to be entered in the blanks marked (a), (b), (c), (d).

cwt.	Quantity		Price per lb. in pence	Cost		
	qr.	lb.		£	s.	d.
1	2	16	$3\frac{1}{2}$		(a)	
2	3	10	(b)	5	19	3
	(c)		$2\frac{1}{4}$	1	9	3
			Total cost -		(d)	

16. Calculate the average daily cost of the business tour : 5 days at 14s. 3d. per day ; 15 days at 12s. 3d. per day ; 3 days at 14s. 10d. per day and 10 days at 13s. 6d. per day.

17. A small undertaking commenced a week's business on October 3rd with £353 in the bank and £5 14s. 6d. in cash. The following transactions were made during the week :

October 3rd : Rent paid out of cash, £5 10s.

„ 4th : Goods purchased and paid for by cheque, £61 18s. 7d.

„ 5th : Cash sales for two days, £58 16s. 5d., of which £50 was paid into the bank.

„ 7th : Payment by cheque for goods purchased, £75 8s. 10d.

„ 8th : Cash sales, £67 13s. 2d., of which £65 was paid into the bank.

Wages paid out of cash, £5 15s. 6d.

Find the balance on the week's business (i) in the bank, and (ii) in cash.

18. A tradesman, during the half-year ending on June 30th, deposited the following amounts in his bank :

£53 2s. 2d. ; £109 18s. 6d. ; £91 13s. 1d. ; £44 6s. 3d.

From these, he made the following payments for goods, etc. :

£23 16s. 9d. ; £28 1s. 11d. ; £19 15s. 4d. ; £30 12s. 11d. ;

£47 12s. 10d. ; £58 15s. 1d.

Find his balance in the bank on June 30th.

19. From a drum containing three-quarters of a mile of wire, there were cut 8 pieces each of length 46 yd. 2 ft., 13 pieces each of length 23 yd. 1 ft. and 17 pieces each of length 13 yd. 2 ft. Find the length of the wire remaining on the drum.

20. If a mile of cable weighs 5 tons 11 cwt. 23 lb., find in tons, cwt., lb., correct to the nearest lb., the weight of 9 miles 825 yd. of cable. (L.Ch.C.)

21. The weight of 97 castings is 17 tons 13 cwt. 1 qr. 12 lb. Find the average weight of each casting and its value if the material costs £1 15s. per cwt.

22. A dealer bought 4 separate lots of coal for which he paid £24 11s. 4d., £22 11s. 11d., £42 13s. 1d., £96 14s. 6d. respectively. The prices per ton were, £1 2s. 4d., £1 6s. 7d., £1 9s. 5d., £1 16s. 6d. respectively. What average price per ton did he pay for the coal as a whole? (U.L.C.I.)

23. When wheat was 44s. per qr. and the yield was 5 qr. per acre, a certain field brought in produce worth £135 13s. 4d. Calculate the value of the crop of the same field when the yield was $3\frac{3}{4}$ qr. per acre and wheat was 48s. per qr. (U.L.C.I.)

24. Find the number of francs equivalent to £1 when a charge of 69 centimes per kilometre on French railways is equal in value to $1\frac{1}{2}$ d. per mile on English railways, taking 5 miles = 8 kilometres.

25. A fruiterer buys 4 bushels of apples at 17s. 2d. per bushel, $5\frac{1}{2}$ bushels at 16s. 4d. per bushel and six bushels at 14s. 10d. per bushel. He sells all the apples at a uniform price of $5\frac{1}{2}$ d. per lb. and makes a profit of £1 16s. 8d. Find the number of lb. equivalent to one bushel.

26. A warehouse contains

	17 cases of goods each weighing 5 cwt. 2 qr. 11 lb.,
	19 cases each weighing 4 cwt. 2 qr. 27 lb.,
	9 " " " 3 " 3 " 17 "
and	12 " " " 4 " 3 " 19 "

The goods are to be re-packed in 57 cases of the same size and, when filled, of the same weight. Calculate the weight of each of these filled cases.

27. A New York merchant exchanged 831 dollars to pay a Paris bill for 30470 francs when the exchange between Paris and London was $178\frac{3}{4}$ francs to the £. Find, in dollars and cents, the exchange rate between New York and London.

28. A United States silver dollar weighs 412·5 grains and is nine-tenths fine. Find its intrinsic value in English money when English standard silver, thirty-seven fortieths fine is selling at 2s. 5d. per ounce Troy of 480 grains. (R.S.A.)

29. Wine is imported from France at an inclusive price of 26 francs 70 centimes per litre. It is sold in London at £1 12s. 6d. per dozen bottles ; calculate the profit made on the sale of 14 dozen bottles, given that 1 litre = 1·75 pints, 1 gallon fills 6 bottles and £1 = 178 francs.

30. Given that a sovereign weighs 7·98805 grams and is eleven-twelfths fine and a dollar weighs 1·6718 grams and is 0·9 fine gold, find

(i) the Mint Par of Exchange in dollars, to three places of decimals, of £1,

(ii) the intrinsic value of a dollar in pence, to one place of decimals.

31. A London trader cashed a New York cheque for 9600 dollars, the rate of exchange being 5·11 dollars to the £. With the proceeds he found he was just able to buy bills to settle two debts, one in Paris for 112,000 francs and the other in Brussels for 12,447 belgas. The rate on Paris was £1 = 83·22 francs ; what was the rate on Brussels? (U.L.C.I.)

CHAPTER IV

DECIMALISATION AND DE-DECIMALISATION

4.1. The Need for Decimalisation.

THE fact that the British system of money, weights and measures is not a decimal one gives rise to much laborious calculation in practical arithmetical operations. To render such work as simple as possible however, sub-units are generally expressed as decimals of the larger and more commonly used units. This process, known as **decimalisation**, has already been introduced in an elementary form in Sections 2.5 and 2.6, but some more rapid and practical method is needed for commercial computation.

4.2. Practical Decimalisation of Money.

Possibly one of the most frequently occurring cases of decimalisation is that concerning money. In Ex. 4 (page 23), it was shewn that, by the ordinary method of division, (i) 7s. 7d. = £0.379 and (ii) 13s. 5d. = £0.671.

Now consider how these and similar results may be obtained in a more rapid and practical manner.

Suppose the sum of money contains S shillings, where S denotes any whole number between 1 and 20.

Since $1 \text{ shilling} = \text{£}\frac{1}{20} = \text{£}0.05$;

$\therefore S \text{ shillings} = \text{£}(0.05 \times S)$;

thus,

$7 \text{ shillings} = \text{£}(0.05 \times 7) = \text{£}0.35$,

$13 \text{ shillings} = \text{£}(0.05 \times 13) = \text{£}0.65$, and so on.

Further, suppose in addition to S shillings, the sum of money contains pence and halfpence or farthings. Reduce this to farthings and represent the number by f ; then f will denote any whole number between 1 and 48.

$$\text{Now} \quad 1 \text{ farthing} = \pounds \frac{1}{960} = \pounds 0\cdot001041666 \dots$$

But, since the farthing is the coin of smallest value, the decimal equivalent need only be taken to three places ;

$$\therefore 1 \text{ farthing} = \pounds 0\cdot001.$$

If $\pounds \frac{1}{1000}$ or $\pounds 0\cdot001$ be called a *mil*, then

$$1 \text{ farthing} = \pounds 0\cdot001 = 1 \text{ mil},$$

$$\begin{aligned} \text{and} \quad 12 \text{ farthings or } 3\text{d.} &= \pounds \frac{1}{80} = \pounds 0\cdot0125 = 12\cdot5 \text{ mils.} \\ 24 \text{ farthings or } 6\text{d.} &= \pounds \frac{1}{40} = \pounds 0\cdot025 = 25 \text{ mils.} \\ 36 \text{ farthings or } 9\text{d.} &= \pounds \frac{3}{80} = \pounds 0\cdot0375 = 37\cdot5 \text{ mils.} \\ 48 \text{ farthings or } 1\text{s.} &= \pounds \frac{1}{20} = \pounds 0\cdot05 = 50 \text{ mils.} \end{aligned}$$

It is evident that, since 1 mil is slightly less than a farthing and the farthing is the coin of least value in circulation, it is impracticable to consider fractions of a mil ; hence, correct to the nearest mil, the equivalent of 12 farthings must be taken as 13 mils, i.e. $(12 + 1)$ mils, and of 36 farthings as 38 mils or $(36 + 2)$ mils.

Therefore, to sum up,

f farthings are equivalent to

- (i) f mils, when f is less than 12,
- (ii) $(f + 1)$ mils, when f lies between 12 and 35 inclusive,
- (iii) $(f + 2)$ mils, when f lies between 36 and 48 inclusive. \clubsuit

By these simple rules, any sum of money may readily be expressed as a decimal of £1 correct to three places. To illustrate the method of application, the amounts quoted above from Ex. 4 page 23, will be decimalised again.

Ex. 1. *Express as decimals of a £, correct to three places,*

- (i) 7s. 7d. and (ii) 13s. 5d.

From the rules already discussed above,

$$\begin{aligned}
 \text{(i)} \quad 7\text{s.} &= £(0\cdot05 \times 7) && = £0\cdot350 \\
 7\text{d.} &= 28 \text{ farthings} = (28 + 1) \text{ mils} = £0\cdot029 \\
 &&& \therefore 7\text{s. } 7\text{d.} = \underline{£0\cdot379} \\
 \text{(ii)} \quad 13\text{s.} &= £(0\cdot05 \times 13) && = £0\cdot650 \\
 5\text{d.} &= 20 \text{ farthings} = (20 + 1) \text{ mils} = £0\cdot021 \\
 &&& \therefore 13\text{s. } 5\text{d.} = \underline{£0\cdot671}
 \end{aligned}$$

After a little practice the student should be able to convert any sum of money into a decimal of a £ mentally.

4·3. The Reverse Process.

When a sum of money is expressed as a decimal of £1, the process of converting the decimal into shillings and pence is sometimes called **de-decimalisation**. In the case of a decimal to three places, or even four, it is quite easy to apply the rules of Section 4·2 in the reverse order.

Ex. 2. *Express in £ s. d. (i) £0·893, (ii) £2·0612 to the nearest penny, and (iii) £4·627, (iv) £0·4618 to the nearest farthing.*

$$\begin{aligned}
 \text{(i)} \quad £0\cdot893 &= £(0\cdot85 + 0\cdot043) = 17\text{s.} + 43 \text{ mils} \\
 &= 17\text{s.} + 41 \text{ farthings} = \underline{17\text{s. } 10\text{d.}} \text{ to the nearest penny.} \\
 \text{(ii)} \quad £2\cdot0612 &= £(2 + 0\cdot05 + 0\cdot0112) = £2 \text{ 1s.} + 11\cdot2 \text{ mils} \\
 &= £2 \text{ 1s.} + 11 \text{ farthings} \\
 &= \underline{£2 \text{ 1s. } 3\text{d.}} \text{ to the nearest penny.} \\
 \text{(iii)} \quad £4\cdot627 &= £(4 + 0\cdot60 + 0\cdot027) = £4 \text{ 12s.} + 27 \text{ mils} \\
 &= \underline{£4 \text{ 12s.} + 26 \text{ farthings}} \\
 &= \underline{£4 \text{ 12s. } 6\frac{1}{2}\text{d.}} \text{ to the nearest farthing.} \\
 \text{(iv)} \quad £0\cdot4608 &= £(0\cdot45 + 0\cdot018) = 9\text{s.} + 10\cdot8 \text{ mils} \\
 &= 9\text{s.} + 11 \text{ mils to the nearest mil,} \\
 &= 9\text{s.} + 11 \text{ farthings} \\
 &= \underline{9\text{s. } 2\frac{3}{4}\text{d.}} \text{ to the nearest farthing.}
 \end{aligned}$$

Note that shillings alone produce finite decimals of a £ which end either in 0 or 5, so that any decimal can easily be separated into the parts giving shillings and mils respectively. After a little practice the conversion may be carried out mentally.

EXERCISES 4A

The answers to the following exercises should be written down without working on paper.

Express each of the following sums of money as a decimal of £1 correct to three places :

- | | |
|-----------------------------------|----------------------------------|
| 1. 18s. 8d. 933 | 2. £3 6s. 10d. (R.S.A.) 3.111 |
| 3. £3 17s. 5d. (R.S.A.) 3.286 | 4. £5 11s. 7d. (R.S.A.) 5.535 |
| 5. 17s. 8½d. 386 | 6. £8 18s. 7½d. 8.875 |
| 7. 7s. 9½d. (R.S.A.) 7.479 | 8. £1 15s. 4¾d. (U.L.C.I.) 1.777 |
| 9. £15 15s. 10¼d. 15.479 | 10. £1 17s. 8½d. (R.S.A.) 1.844 |
| 11. 18s. 9¾d. 18.479 | 12. 17s. 10¼d. (U.L.C.I.) 1.844 |
| 13. £3 15s. 5½d. (R.S.A.) 3.286 | 14. £21 13s. 5¾d. 21.286 |
| 15. 12s. 4½d. 12.479 | 16. 6s. 3¾d. 6.578 |
| 17. £13 14s. 9¼d. 13.286 | 18. 18s. 5½d. 18.286 |
| 19. £6 1s. 10¼d. (U.L.C.I.) 6.179 | 20. £1 1s. 2¾d. 1.042 |

Convert each of the following into £ s. d. correct to the nearest penny :

- | | |
|---------------------------------|-----------------------|
| 21. £5.288. (U.L.C.I.) | 22. £0.547. (R.S.A.) |
| 23. £3.527. 3 10 6 (R.S.A.) | 24. £0.083. 8 3 4 |
| 25. £6.795. 6 15 10½ (U.L.C.I.) | 26. £0.678. 6 15 10½ |
| 27. £0.5439. 5 4 3 9 | 28. £3.1207. 3 12 0 7 |
| 29. £15.3458. 15 6 10 8 | 30. £4.7286. 4 14 5 4 |

Express each of the following decimals of a £ in £ s. d. to the nearest farthing :

- | | |
|------------------------|------------------------|
| 31. £0.068. (U.L.C.I.) | 32. £0.891. (U.L.C.I.) |
| 33. £5.876. 5 17 5 2 | 34. £13.608. 13 12 1 6 |
| 35. £1.322. (U.L.C.I.) | 36. £1.578. (U.L.C.I.) |
| 37. £0.7483. 7 9 8 3 | 38. £2.8735. 2 17 6 1 |
| 39. £1.0806. 1 1 6 1 | 40. £7.9817. 7 19 7 1 |

4.4. Decimalisation of Money to Seven Places.

When sums of money have to be multiplied by large numbers, as in the calculation of costs, the decimalisation must be taken to at least seven places. For this purpose it is usual to have recourse to a table of decimalisation ; such a table is given at the end of the book on page 334. The following example will show the necessity of decimalisation to more than three places.

Ex. 3. Express 1s. $6\frac{1}{4}$ d. as a decimal of £1 correct to (i) three places, and (ii) eight places.

Hence, find the error by using (i) in calculating the value of 8624 rupees when one rupee is equivalent to 1s. $6\frac{1}{4}$ d.

(i) By the rules of Section 4.2,

$$\begin{aligned} 1\text{s. } 6\frac{1}{4}\text{d.} &= £0.05 + 25 \text{ farthings} = £0.05 + 26 \text{ mils} \\ &= £0.076 \text{ correct to three places.} \end{aligned}$$

(ii) By the ordinary method of division,

$$1\text{s. } 6\frac{1}{4}\text{d.} = £\frac{73}{960}$$

$$\begin{array}{r|l} 8 & 7.3000 \\ 4 & 0.9125 \\ 3 & 0.228125 \\ \hline & 0.076041666\dots \end{array}$$

$$= £\frac{7.3}{96} = £0.07604167, \text{ correct to eight places.}$$

Generally in practice, decimalisation tables would be used. Thus, from the table on page 334,

$$\begin{aligned} 1\text{s.} &= £0.05 \\ 6\frac{1}{4}\text{d.} &= 25 \text{ farthings} = £0.02604167 \\ \therefore 1\text{s. } 6\frac{1}{4}\text{d.} &= £0.07604167 \end{aligned}$$

Now, to find the value of 8624 rupees at 1s. $6\frac{1}{4}$ d. each, from (i) the value of 8000 rupees = $£0.076 \times 8000 = £608.0000$

„	„	600	„	=	„	×	600	=	£	45.6000
„	„	20	„	=	„	×	20	=	£	1.5200
„	„	4	„	=	„	×	4	=	£	0.3040
∴	„	„	8624	„	=					<u>£655.4240</u> = £655 8s. 6d.

From (ii), taking the decimal value to seven places and retaining only four places in the products,

$$\begin{array}{rclcl}
 \text{the value of } 8000 & = & £0.07604167 \times 8000 & = & £608.3334 \\
 \text{,, } & \text{,,} & 600 = & \text{,,} & \times 600 = £ 45.6250 \\
 \text{,, } & \text{,,} & 20 = & \text{,,} & \times 20 = £ 1.5208 \\
 \text{,, } & \text{,,} & 4 = & \text{,,} & \times 4 = £ 0.3042 \\
 \therefore \text{ the value of } 8624 \text{ rupees} & = & \underline{£655.7834} \\
 & & & & = £655 \text{ 15s. 8d}
 \end{array}$$

To test these values,

$$\begin{array}{rcl}
 8624 \text{ at } 1\text{s.} & = & £431 \quad 4\text{s.} \quad \text{Od.} \\
 \text{,, } 6\text{d.} & = & £215 \quad 12\text{s.} \quad \text{Od.} \\
 \text{,, } \frac{1}{4}\text{d.} & = & £ \quad 8 \quad 19\text{s.} \quad 8\text{d.}
 \end{array}$$

$$\text{Hence, } 8624 \text{ at } 1\text{s. } 6\frac{1}{4}\text{d.} = \underline{£655 \quad 15\text{s.} \quad 8\text{d.}}$$

It is therefore evident that, in taking the decimalisation to three places only, the result is inaccurate ; in this case it is as much as 7s. 2d. short, whilst to eight places the correct value is obtainable.

As a general rule, when the multiplier is not greater than 10^n , where n denotes any integer, the decimalisation must be taken to $(3+n)$ places ; thus, for a multiplier not exceeding

$$\begin{array}{lll}
 10^1 \text{ or } 10, & \text{decimalisation to } 3+1, \text{ or } 4 \text{ places, is needed,} \\
 10^2 \text{ or } 100, & \text{,, } & \text{,, } 3+2, \text{ or } 5 \quad \text{,, } \text{,, } \text{,,} \\
 10^3 \text{ or } 1000, & \text{,, } & \text{,, } 3+3, \text{ or } 6 \quad \text{,, } \text{,, } \text{,,} \\
 10^4 \text{ or } 10000, & \text{,, } & \text{,, } 3+4, \text{ or } 7 \quad \text{,, } \text{,, } \text{,,}
 \end{array}$$

and so on.

It follows therefore that eight places will be sufficient for most practical purposes.

4.5. Tables of Nine Multiples.

Where the cost of any number of articles at a uniform price is frequently needed, tables are made giving the costs in decimals of a £ of 1, 2, 3, ... 9 articles. These are known as **Tables of Nine Multiples**.

Ex. 4. Construct a table of Nine Multiples correct to seven places for a price of 13s. 8½d. Use the table to find the cost of 5374 articles at this price.

$$13\text{s. } 8\frac{1}{2}\text{d.} = \pounds \frac{329}{480} = \pounds \frac{32\cdot9}{48}$$

= £0·68541667 to eight places.

$$\begin{array}{r|l} 4 & 32\cdot9 \\ 4 & 8\cdot225 \\ 3 & 2\cdot05625 \\ \hline & 0\cdot68541666\ldots \end{array}$$

Otherwise, from the table on page 334,

$$13\text{s.} \qquad \qquad \qquad = \pounds 0\cdot65$$

$$8\frac{1}{2}\text{d.} = 34 \text{ farthings} = \pounds 0\cdot0354167 \text{ correct to seven places.}$$

$$\therefore 13\text{s. } 8\frac{1}{2}\text{d.} = \pounds 0\cdot6854167 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

Note that the decimal must be taken to 8 places to ensure accuracy to 7 places.

The accompanying table may now readily be made up.

To find the cost of 5374 articles at 13s. 8½d. each from the table, only four places need be retained in the products.

Hence, correct to four places, the cost of

$$5000 = \pounds 3\cdot4270833 \times 1000 = \pounds 3427\cdot0833$$

$$300 = \pounds 2\cdot0562500 \times 100 = \pounds 205\cdot6250$$

$$70 = \pounds 4\cdot7979167 \times 10 = \pounds 47\cdot9792$$

$$4 \qquad \qquad \qquad = \pounds 2\cdot7417$$

$$\therefore \text{ the cost of 5374 articles} = \pounds 3683\cdot4292 = \pounds 3683 \text{ 8s. 7d.}$$

Cost at 13s. 8½d.	
1	£0·6854167
2	£1·3708333
3	£2·0562500
4	£2·7416667
5	£3·4270833
6	£4·1125000
7	£4·7979167
8	£5·4833333
9	£6·1687500

4·6. Approximate Decimalisation of Weight.

The most frequently required decimalisation of weight is that of expressing cwt., qr. and lb. as decimals of a ton. Since 1 ton = 20 cwt., cwt. may readily be decimalised in the same way as shillings, viz. by multiplying the number of cwt. by 0·05.

For quarters, since 1 qr. = $\frac{1}{8}$ ton = 0·125 ton,

$$\therefore 2 \text{ quarters} = (0\cdot125 \times 2) \text{ ton} = 0\cdot25 \text{ ton, and}$$

$$3 \text{ quarters} = (0\cdot125 \times 3) \text{ ton} = 0\cdot375 \text{ ton.}$$

Finally, for lb., a rapid method of decimalisation is not quite so simple as in the case of farthings. An approximate rule may, however, be used which will express lb. in decimals of a ton correct to four places.

Let $\frac{1}{10000}$ of a ton be called a **decimilt**, i.e. a *deci-mil* of a ton. This new unit will be generally denoted in the abbreviated form, *dmt.*

Now $1 \text{ lb.} = \frac{1}{2240} \text{ ton} = 0.00044643 \text{ ton} = 4.5 \text{ decimilts nearly.}$

From this fact, the following rule will give lb. as decimals of a ton correct to three places.

Taking 1 decimilt as 0.0001 ton,

1 lb. is equivalent to 4.5 decimilts, correct to five places ;

w lb. are equivalent to

- (i) $(w \times 4.5)$ decimilts when w lies between 2 and 14 inclusive, provided the 0.5 dmt. in products for ODD values of w is ignored;
- (ii) $\frac{1}{2}(9w - 1)$ decimilts when w lies between 15 and 27 inclusive, provided the 0.5 dmt. in products for EVEN values of w is ignored.

Ex. 5. Express 8 cwt. 3 qr. 22 lb. as a decimal of 1 ton, correct to four places.

First convert the lb. by applying the above rule.

Since 22 lies between 15 and 27, $22 \text{ lb.} = \frac{1}{2}(9 \times 22 - 1) \text{ dmt.}$

$= 98 \text{ dmt., ignoring the } 0.5 \text{ dmt. in the product.}$

Hence $22 \text{ lb.} = 98 \text{ dmt.} = 0.0098 \text{ ton}$

$3 \text{ qr.} = 3 \times 0.0125 \text{ ton} = 0.0375 \text{ ton}$

$8 \text{ cwt.} = 8 \times 0.05 \text{ ton} = 0.4000 \text{ ton}$

\therefore by addition, $8 \text{ cwt. } 3 \text{ qr. } 22 \text{ lb.} = \overline{0.4473} \text{ ton.}$

To check this result by the ordinary method of division,

$8 \text{ cwt. } 3 \text{ qr. } 22 \text{ lb.} = (896 + 84 + 22) \text{ lb.} = 1002 \text{ lb.}$

$\therefore 8 \text{ cwt. } 3 \text{ qr. } 22 \text{ lb.} = \frac{1002}{2240} \text{ ton}$

$= \frac{100.2}{224} \text{ ton}$

$$\begin{array}{r|l} 8 & 100.2 \\ 4 & \underline{12.525} \\ 7 & \underline{3.13125} \\ & 0.447321 \dots \end{array}$$

$= 0.4473 \text{ ton, correct to four places.}$

4.7. Decimalisation to Nine Places.

As in the case of money, when weights or lengths have to be multiplied frequently by large numbers, it is essential to decimalise to at least nine places. For this purpose tables are often used and such tables are provided on pages 335-7 by which cwt., qr., lb. can be readily converted into decimals of a ton and yards into decimals of a mile to nine places. The following examples will show how the tables may be applied.

Ex. 6. *Express, by use of tables,*

(i) 7 cwt. 3 qr. 11.8 lb. as a decimal of a ton,

(ii) 1487.6 yards as a decimal of a mile.

Give each result correct to nine places; hence write down the decimals correct to four places.

(i) From the table on page 335,

	ton
7 cwt. =	0.35
3 qr. =	0.0375
11 lb. =	0.004910714
0.8 lb. = $\frac{1}{10}$ of 8 lb. =	0.0003571429
\therefore 7 cwt. 3 qr. 11.8 lb. =	<u>0.3927678569</u>
	= 0.392767857 ton, correct to nine places
	= 0.3928 ton, correct to four places.

(ii) From the table on page 337,

	Mile
1400 yards =	0.795454545
87 yards =	0.049431818
0.6 yard = $\frac{1}{10}$ of 6 yd. =	0.0003409091
\therefore 1487.6 yards	<u>= 0.8452272721</u> mile, correct to nine places
	= 0.8452 mile, correct to four places.

Ex. 7. *Find, by decimalisation, the total weight of 2465 cases of goods each weighing 3 cwt. 2 qr. 18 lb.*

First decimalise 3 cwt. 2 qr. 18 lb.

From the table on page 335,

$$\begin{array}{rcl}
 3 \text{ cwt.} & = & 0\cdot15 \quad \text{ton.} \\
 2 \text{ qr.} & = & 0\cdot025 \quad \text{,,} \\
 18 \text{ lb.} & = & 0\cdot008035714 \quad \text{,,} \\
 \hline
 \therefore 3 \text{ cwt. 2 qr. 18 lb.} & = & 0\cdot183035714 \quad \text{,,}
 \end{array}$$

Now, in calculating the total weight of the cases, it is only necessary to retain four places, since the smallest unit involved is a lb.

Hence the weight of

$$\begin{array}{rcl}
 2000 \text{ cases} & = & 0\cdot183035714 \times 2000 \text{ tons} = 366\cdot0714 \text{ tons.} \\
 400 \text{ ,,} & = & \text{,,} \times 400 \text{ ,,} = 73\cdot2143 \text{ ,,} \\
 60 \text{ ,,} & = & \text{,,} \times 60 \text{ ,,} = 10\cdot9821 \text{ ,,} \\
 5 \text{ ,,} & = & \text{,,} \times 5 \text{ ,,} = 0\cdot9152 \text{ ,,} \\
 \hline
 \therefore 2465 \text{ cases will weigh} & & 451\cdot1830 \text{ tons}
 \end{array}$$

$$= 451 \text{ tons 3 cwt. 2 qr. 18 lb. from the table ;}$$

$$\therefore \text{ total weight} = 451 \text{ tons 3 cwt. 2 qr. 18 lb.}$$

If the weights of varying numbers of similar cases had to be found frequently, a table of nine multiples would first be made.

The weight of	cwt.	qr.	lb.	tons
1 case =	3	2	18	$= 0\cdot183035714$
2 cases =	7	1	8	$= 0\cdot366071429$
3 ,, =	10	3	26	$= 0\cdot549107143$
4 ,, =	14	2	16	$= 0\cdot732142857$
5 ,, =	18	1	6	$= 0\cdot915178571$
6 ,, =	21	3	24	$= 1\cdot098214286$
7 ,, =	25	2	14	$= 1\cdot281250000$
8 ,, =	29	1	4	$= 1\cdot464285714$
9 ,, =	32	3	22	$= 1\cdot647321429$

With the use of this table, the weight of any number of cases may be found ; thus for 2465 :

Weight of	Tons	Tons
2000 cases	$= 0.366071429 \times 1000$	$= 366.0714$
400 „	$= 0.732142857 \times 100$	$= 73.2143$
60 „	$= 1.098214286 \times 10$	$= 10.9821$
5 „		$= 0.9152$
2465		$= 451.1830$

i.e. the total weight of 2465 cases is 451.1830 tons or

451 tons 3 cwt. 2 qr. 18 lb. as before.

It will be noticed that the four lines in each solution are identical.

Finally, it may sometimes be convenient to work as follows for single calculations.

	cwt.		
	2465	= weight at 1 cwt. each	
	3		
	7395	= „ „ 3 cwt.	
2 qr. $= \frac{1}{2}$ cwt.	1232.5	= „ „ 2 qr.	
14 lb. $= \frac{1}{4}$ of 2 qr.	308.125	= „ „ 14 lb.	
$3\frac{1}{2}$ lb. $= \frac{1}{4}$ of 14 lb.	77.03125	= „ „ $3\frac{1}{2}$ lb.	
$\frac{1}{2}$ lb. $= \frac{1}{7}$ of $3\frac{1}{2}$ lb.	11.00446	= „ „ $\frac{1}{2}$ lb.	
	9023.66071	= „ „ 3 cwt. 2 qr. 18 lb.	

Hence, the total weight of 2465 cases

$= 9023.6607$ cwt.

$= 451$ tons 3.6607 cwt.

$= 451$ tons 3 cwt. 2 qr. 18 lb.

as already found in the two previous solutions.

4.8. A Common Type of Problem.

The method of decimalisation may often be conveniently applied to the calculation of costs and similar problems where the values are required of lower units than those for which a price is quoted. The following example will show how the method may be used in such cases.

Ex. 8. Find the cost, to the nearest penny, of 47 tons 18 cwt. 3 qr. 13 lb. of material at £7 13s. 10d. per ton.

First decimalise 18 cwt. 3 qr. 13 lb.

$$\begin{array}{rcl}
 18 \text{ cwt.} & = 0.90 & \text{ton.} \\
 3 \text{ qr.} & = 0.0375 & \text{,,} \\
 13 \text{ lb.} & = 0.005803571 & \text{,,} \\
 \hline
 \therefore 18 \text{ cwt. 3 qr. 13 lb.} & = 0.943303571 & \text{,,}
 \end{array}$$

Hence, correct to four places,

$$47 \text{ tons 18 cwt. 3 qr. 13 lb.} = 47.9433 \text{ tons.}$$

Now proceed as follows :

	£	
	47.9433 = cost at £1 per ton.	
	7	
	335.6031 =	,, £7
10s. = £ $\frac{1}{2}$	23.9717 =	,, 10s.
2s. = $\frac{1}{5}$ of 10s.	4.7943 =	,, 2s.
1s. = $\frac{1}{2}$ of 2s.	2.3972 =	,, 1s.
6d. = $\frac{1}{2}$ of 1s.	1.1986 =	,, 6d.
4d. = $\frac{1}{3}$ of 1s.	0.7991 =	,, 4d.
	368.7640 =	,, £7 13s. 10d. per ton.

Hence, the required cost = £368.764

= £368 15s. 3d. to the nearest penny.

EXERCISES 4B

By the use of the tables, express the following sums of money as decimals of £1, correct to *seven* places. Write down also each result correct to *four* places.

- | | | |
|-------------------------------|------------------------------|--------------------------------|
| 1. 11s. 11d. | 2. 17s. 5d. | 3. £1 3s. 7d. |
| 4. £8 19s. 1d. | 5. £5 15s. 11d. | 6. 13s. 5 $\frac{3}{4}$ d. |
| 7. 19s. 2 $\frac{1}{2}$ d. | 8. £2 1s. 7 $\frac{1}{4}$ d. | 9. £3 11s. 10 $\frac{3}{4}$ d. |
| 10. £7 0s. 9 $\frac{1}{4}$ d. | | |

Using the rule of Section 4·6, express the following as decimals of a ton, correct to *three* places.

- | | |
|------------------------|--------------------------------|
| 11. 6 cwt. 13 lb. | 12. 4 cwt. 1 qr. 8 lb. |
| 13. 327 lb. | 14. 3 qr. 21 lb. |
| 15. 9 cwt. 2 qr. 8 lb. | 16. 1 ton 13 cwt. 1 qr. 19 lb. |

Express as decimals of a ton, correct to *nine* places, the following by using the appropriate table. Write down each result also as a decimal to *five* places.

- | | |
|---------------------------------|-----------------------------------|
| 17. 17 cwt. 3 qr. 23 lb. | 18. 5 cwt. 97 lb. |
| 19. 185 lb. | 20. 2 tons 1 qr. 9 lb. |
| 21. 8 tons 11 cwt. 3 qr. 26 lb. | 22. 2 qr. 15 lb. 4 oz. |
| 23. 3 cwt. 1 qr. 17·3 lb. | 24. 3 tons 19 cwt. 2 qr. 19·8 lb. |
| 25. 7 tons 13 cwt. 62·7 lb. | 26. 17 cwt. 2 qr. 13 lb. 6 oz. |

Convert the following lengths into decimals of a mile, correct to *nine* places by use of the table. Write down each result also to *five* places.

- | | |
|---------------------------|---------------------------------|
| 27. 1373 yards. | 28. 739 yards. |
| 29. 817·6 yards. | 30. 2 miles 357·2 yards. |
| 31. 34 chains 70 links. | 32. 67 chains 53 links. |
| 33. 83 chains 13·4 yards. | 34. 4 miles 18 chains 83 links. |

In the following exercises the method of decimalisation should be used wherever convenient.

35. The following table gives the price of any number of articles from 1 to 9 at 11s. 10½d. each :

Price of 1 article	=	£0·59375
„ 2 articles	=	£1·18750
„ 3 „	=	£1·78125
„ 4 „	=	£2·37500
„ 5 „	=	£2·96875
„ 6 „	=	£3·56250
„ 7 „	=	£4·15625
„ 8 „	=	£4·75000
„ 9 „	=	£5·34375

Using the table, find the prices of

- (i) 47 articles, (ii) 189 articles, (iii) 8250 articles.

(U.L.C.I.)

36. Construct a table of nine multiples for a price of 1s. $2\frac{1}{4}$ d., and use it to determine the cost of (i) 16 dozen, (ii) 9 gross, and (iii) 5874 articles at this price.

37. Make out a table of nine multiples for a price of 2s. $10\frac{1}{2}$ d., and use it to find the cost of (i) 463, (ii) 1874 and (iii) 5697 articles at this price.

38. By first making a table of nine multiples, find the cost of 78 dozen articles at 4s. $7\frac{1}{2}$ d. each.

39. Construct a table of nine multiples giving the price per lb. of a commodity which costs £25 17s. 4d. per ton. Hence determine to the nearest penny, the cost of 3 tons 8 cwt. 2 qr. 19 lb. of the commodity.

40. The rateable value of the property in a certain town is £5,611,065 and the rate levied for a certain purpose is 1s. $1\frac{3}{4}$ d. in the £. Calculate, to the nearest penny, the total amount raised for the purpose of this rate. (R.S.A.)

41. Find the price obtained for 19,516 acres of land sold at an average of £6 18s. 6d. per acre. (R.S.A.)

42. Find, to the nearest penny, the cost of 73 tons 17 cwt. 93 lb. at £5 12s. 9d. per ton. (L.Ch.C.)

43. Calculate, to the nearest penny, the cost of 54 tons 16 cwt. 3 qr. 19 lb. at £6 4s. 10d. per ton.

44. Find, to the nearest farthing, the cost of 37 tons 17 cwt. 3 qr. 25 lb. at £6 15s. 4d. per ton.

45. Decimalise 2s. $4\frac{3}{4}$ d. ; hence find the cost of 50 tons of tea at 2s. $4\frac{3}{4}$ d. per lb.

46. Express 13 dwt. 11·8 grains as a decimal of 1 ounce Troy, correct to seven places. Hence calculate, to the nearest penny, the value of 5 oz. 13 dwt. 11·8 gr. of pure gold at £7 6s. 1d. per oz.

47. Find the import duty on 387 cwt. 2 qr. 15 lb. of sugar at 6s. 4·6d. per cwt. (R.S.A.)

48. Calculate the weight of 3573 crates of goods, each weighing 2 cwt. 2 qr. 13·5 lb.

49. Calculate, to the nearest penny, the cost of 18 miles 1267 yd. of telegraph wire at £8 13s. 3d. per mile.

50. Calculate, to the nearest lb., the weight of 7 miles 928 yd. of cable, taking the weight of one mile as 5 tons 13 cwt. 17 lb.

51. In an examination a candidate had to find the value of 7500 rupees at 1s. $6\frac{1}{4}$ d. per rupee. Misusing the method of decimalisation of money, he wrote down 1s. $6\frac{1}{4}$ d. as the decimal of £1 correct to *three* places and multiplied by 7500. Find the error in his answer. (R.S.A.)

52. Find in feet the error made when converting 1528 metres into feet by taking

- (i) 11 metres equivalent to 12 yards,
- (ii) 8 kilometres equivalent to 5 miles.

The value of a metre correct to 5 places is 39·37079 inches.

(R.S.A.)

CHAPTER V

RATIO, PROPORTION, PERCENTAGE AND AVERAGE

5.1. Ratio.

The method of comparing two magnitudes of the same kind has already been dealt with in Sections 1.3 and 4.2, where it has been shown how one magnitude may be expressed as a fraction, ordinary or decimal, of the other, provided both are in the same units. Another name for such a fraction is **ratio**.

Let two quantities be expressed in terms of the same unit and suppose m , n denote the respective numbers of those units; then the ratio of m to n is written in the form $m : n$ and described in the words *as m is to n* . Thus $m : n$ is only another way of writing $\frac{m}{n}$.

The alloy of which an 18-carat gold article is made usually consists of pure gold and copper. In 24 parts by weight of the alloy, 18 are pure gold and 6 copper, so that the ratio of the weights of copper to gold is $6 : 18 = \frac{6}{18} = \frac{1}{3}$. The composition of the alloy may also be expressed in terms of the ratio of the weights of pure gold to that of the alloy. This is $18 : 24 = \frac{18}{24} = \frac{3}{4}$. For this reason, 18-carat gold is sometimes described as $\frac{3}{4}$ fine. Similarly, 22-carat gold is $\frac{11}{12}$ fine.

Ex. 1. *Express the ratio of a U.S. short ton to a British ton in its simplest form, a short ton being equivalent to 17 cwt. 3 qr. 12 lb.*

First convert each ton to lb., thus

$$\text{a U.S. short ton} = (17 \times 112 + 3 \times 28 + 12) \text{ lb.} = 2000 \text{ lb.}$$

$$\text{and} \quad \text{a British ton} = 2240 \text{ lb.}$$

$$\therefore \text{required ratio} = 2000 : 2240 = \frac{2000}{2240} = \frac{25}{28} = \mathbf{25 : 28}.$$

Ex. 2. *A and B enter into partnership, A providing £4914 and B £6048 as capital. At the end of a year, the net profit made was £1479; how much of this is respectively due to A and to B?*

It is evident that the profit must be shared by the two partners in the ratio of their invested capital;

$$\therefore A's \text{ capital} : B's \text{ capital} = 4914 : 6048,$$

or in the more convenient fractional form

$$\frac{A's \text{ capital}}{B's \text{ capital}} = \frac{4914}{6048} = \frac{13}{16},$$

on reducing to lowest terms.

Hence, of the profit, 13 parts should go to A and 16 to B, so that the profit must be divided into 13 + 16 or 29 parts.

$$\therefore A's \text{ share} = \frac{13}{29} \text{ of } £1479 = £(13 \times 51) = £663.$$

$$\text{and} \quad B's \text{ share} = \frac{16}{29} \text{ of } £1479 = £(16 \times 51) = £816.$$

5-2. Symbolical Representation of a Number.

In very many cases it is convenient to denote an unknown number symbolically by a letter. The answer to a problem may thus be represented and then a simple relation obtained from which the actual value denoted by the symbol may be found. The method will now be frequently used and the following examples will illustrate how it may be effectively applied.

Ex. 3. *How many pieces, each $4\frac{3}{4}$ yards long, can be cut from a roll of material containing $85\frac{1}{2}$ yards?*

Suppose the required number of pieces be denoted by n , then, since each piece is $4\frac{3}{4}$ yards long, the total length of n pieces will be $(4\frac{3}{4} \times n)$ yards;

$$\text{Hence,} \quad 4\frac{3}{4} \times n = 85\frac{1}{2},$$

so that

$$n = 85\frac{1}{2} \div 4\frac{3}{4} = \frac{171}{2} \times \frac{4}{16} = 9 \times 2 = 18.$$

$$\therefore \text{the required number of lengths} = 18.$$

Ex. 4. *A local authority levies a rate of 8s. 6½d. in the £ on its rateable property and the amount thus raised is £483,021. The next year, a sum of £494,802 is required; how much in the £ must the rate be raised?*

First find the new rate; suppose it is x shillings in the £, then the rateable property = $£(494802 \times 20 \div x)$.

But since a rate of 8s. 6½d. or $8\frac{13}{4}$ shillings in the £ produces £483,021;

$$\therefore \text{the rateable property} = £(483021 \times 20 \div 8\frac{13}{4}).$$

$$\text{Hence} \quad \frac{494802 \times 20}{x} = \frac{483021 \times 20}{8\frac{13}{4}} = \frac{483021 \times 20 \times 24}{205};$$

\therefore by cross multiplication,

$$483021 \times 20 \times 24 \times x = 494802 \times 20 \times 205,$$

$$\text{from which} \quad x = \frac{494802 \times 20 \times 205}{483021 \times 20 \times 24} = \frac{35}{4} = 8\frac{3}{4},$$

on cancelling down.

$$\therefore \text{the new rate} = 8\frac{3}{4} \text{ shillings} = 8\text{s. } 9\text{d.}$$

Hence the increase in the rate per £ = 8s. 9d. - 8s. 6½d. = 2½d.

It may be mentioned that, to illustrate the method of this Section, the amounts have been chosen to give an exact result, but this is very rarely the case in practice.

5.3. Proportion.

When two ratios are equal, the four numbers forming them are said to be **in proportion**; thus, 2, 5, 14, 35 are in proportion, since $\frac{2}{5} = \frac{14}{35}$. This equality is sometimes written in the form

$$2 : 5 = 14 : 35,$$

in which the end numbers, 2, 35, are **extremes** and the numbers between them, 5, 14, are known as **means**. Now the product of the extremes = $2 \times 35 = 70$ and the product of the means = $5 \times 14 = 70$, so that the two products are equal. This is true of all numbers in proportion.

Again, when the means are equal, as in $4 : 6 = 6 : 9$, each of the equal numbers is said to be a **mean proportional** to the others ; thus 6 is the mean proportional to 4 and 9.

Further, for any ratio like $\frac{2}{5}$, there will be an unlimited number of ratios equal to it ; e.g. $\frac{166}{415} = \frac{38}{95} = \frac{34}{85} = \frac{14}{35} = \dots = \frac{2}{5} = 0.4$.

Hence, when several ratios are equal, each single ratio is equal to a fixed number, and when two quantities vary in this manner, one is said to be proportional to the other. For instance, when the price per lb. of some commodity is the same for any weight, the cost of any number of lb. is proportional to that number of lb., since the ratio of the cost to the weight in lb. is always the same, this being the cost of 1 lb.

Hence, summarising these facts symbolically :

(a) Four numbers a, b, m, n are in proportion if

$$a : b = m : n, \quad \text{or} \quad \frac{a}{b} = \frac{m}{n},$$

a, n being the extremes and b, m the means.

(b) The product of the extremes = the product of the means, or

$$a \times n = b \times m.$$

(c) If $a : c = c : n$, then c is a mean proportional to a and n and $c^2 = a \times n$.

(d) When one variable quantity x is proportional to another variable quantity y , then the ratio $\frac{x}{y}$ is constant, i.e. always a fixed number.

Ex. 5. (i) Four numbers 10.4, n , 29.9, 39.1 are in proportion ; find the value of n .

(ii) A quantity M is proportional to another quantity R . When $M = 9.1$, $R = 11.9$; find the value of M when $R = 5.1$.

(i) Since the numbers are in proportion,

$$10.4 : n = 29.9 : 39.1.$$

Hence, from (b) above,

$$29.9 \times n = 10.4 \times 39.1,$$

so that

$$n = \frac{10.4 \times 39.1}{29.9} = \frac{104 \times 391}{299 \times 10} = \frac{136}{10};$$

$$\therefore n = 13.6.$$

(ii) Since M is proportional to R ,

$\therefore \frac{M}{R}$ = a constant value for corresponding pairs of values of M and R .

$$\text{Hence,} \quad \frac{M}{5.1} = \frac{9.1}{11.9}$$

$$\text{or} \quad M = \frac{9.1 \times 5.1}{11.9} = \frac{91 \times 51}{119 \times 10} = \frac{39}{10},$$

$$\text{i.e.} \quad M = 3.9.$$

5.4. Percentage.

In practice it is often necessary to compare unequal ratios, and to do this conveniently some numerical standard of reference is needed. As an example of this necessity, consider the following simple problem.

Ex. 6. *For cash payment a tradesman A takes off $1\frac{1}{2}d.$ in the shilling whilst a tradesman B deducts $2s. 4d.$ in the £. Which are the better terms to the customer?*

$$\text{The ratio of } 1\frac{1}{2}d. \text{ to } 1s. = \frac{1\frac{1}{2}}{12} = \frac{3}{24} = \frac{1}{8}$$

$$\text{and the ratio of } 2s. 4d. \text{ to } £1 = \frac{2\frac{1}{3}}{20} = \frac{7}{60}.$$

Here then it is required to find which of the ratios $\frac{1}{8}$ or $\frac{7}{60}$ is the greater. This may be done by expressing each fraction with the same denominator.

Now the L.C.M. of 8 and 60 = 120 ;

$$\therefore \frac{1}{8} = \frac{15}{120} \quad \text{and} \quad \frac{7}{60} = \frac{14}{120},$$

so that $\frac{1}{8}$ is the larger,

i.e. the better terms are $1\frac{1}{2}d.$ in the shilling.

To find the L.C.M. in every case would be very inconvenient and,

as a consequence, 100 is chosen by which ratios may be readily compared ;

thus, since $\frac{1}{8}$ of 100 = $12\frac{1}{2}$ and $\frac{7}{60}$ of 100 = $11\frac{2}{3}$;

$$\therefore \frac{1}{8} = \frac{12\frac{1}{2}}{100} \quad \text{and} \quad \frac{7}{60} = \frac{11\frac{2}{3}}{100}.$$

In actual practice, the denominator 100 is not written ;

$$\frac{12\frac{1}{2}}{100} \text{ is described as } 12\frac{1}{2} \text{ per cent. ; } \frac{11\frac{2}{3}}{100} \text{ as } 11\frac{2}{3} \text{ per cent.}$$

Per cent. comes from the Latin *per centum*, meaning “ by the hundred ”, and the symbol % is used to denote per cent. ; thus $12\frac{1}{2}$ per cent. is written $12\frac{1}{2}\%$.

By this method it is easy to see which are the better terms quoted in Ex. 6, since

$$1\frac{1}{2}\text{d. in the shilling} = 12\frac{1}{2}\%$$

and

$$2\text{s. } 4\text{d. in the } \pounds = 11\frac{2}{3}\%.$$

Ex. 7. Express each of the following as a percentage : (i) $\frac{11}{16}$, (ii) 0.534, (iii) 5s. $11\frac{1}{2}\text{d.}$ of 18s. 4d., (iv) 51743 of 348952, giving this result correct to one place of decimals.

$$(i) \frac{11}{16} = \frac{11}{16} \times 100 \text{ per cent.} = \frac{275}{4} \text{ per cent.} = 68\frac{3}{4}\%.$$

$$(ii) 0.534 = \frac{534 \times 100}{1000} \text{ per cent.} = \frac{534}{10} \text{ per cent.} = 53.4\%.$$

(iii) Reducing each sum of money to pence,

$$5\text{s. } 11\frac{1}{2}\text{d.} = 71.5 \text{ pence and } 18\text{s. } 4\text{d.} = 220 \text{ pence.}$$

$$\therefore \text{required percentage} = \frac{71.5 \times 100}{220} \text{ per cent.} = \frac{65}{2} \text{ per cent.} \\ = 32.5\%.$$

$$(iv) 51743 \text{ of } 348952 = \frac{51743 \times 100}{348952} \% = \frac{5174300}{348952} \% = 14.8\%.$$

There should be no difficulty in finding percentage if it is remembered that the number a expressed as a fraction of another number

(ii) Again,

value of imported tobacco in 1936 = £18,538,114

“ “ “ “ “ 1935 = £17,576,527

∴ increase in value = $\frac{£961,587}{17576527}$

Hence, increase per cent. = $\frac{96158700}{17576527} = \frac{9616}{1758}$, by Section 2-10,

which, on division, gives 5.47.

∴ required increase = 5.47%.

5-6. Percentage applied to Profit and Loss.

In considering this important application of percentage, it is necessary first to explain briefly some commonly used abbreviations and terms.

- (i) **C.P.** denotes Cost Price.
- (ii) **S.P.** denotes Selling Price.
- (iii) **Prime Cost** means the cost of production exclusive of overhead costs. See (vii) below.
- (iv) **Gross Profit.** The difference between the selling price and the cost price before any necessary expenses in connection with the trading have been paid, the selling price being obviously the greater.
- (v) **Net Profit.** The remainder of the gross profit after all the necessary trading costs have been deducted.
- (vi) **Discount.** The deduction made from the selling price generally for prompt cash payment. (See Chap. VII.)
- (vii) **Overhead Expenses.** Essential costs connected with the business transactions and which do not vary with the fluctuations of trade.
- (viii) **Turnover.** The name given to the cash value of the sales during a given period.

In calculating the percentage profit in a business undertaking, it is more convenient to take the selling price, i.e. the turnover as the

basis. Although it may be more logical to take the cost price, difficulties sometimes arise in ascertaining the precise cost owing to the many variable expenses which often have to be taken into account.

With this introductory explanation, the following worked-out examples should be easily understood.

Ex. 9. *Some goods were bought for £3 13s. 4d. and sold for £3 17s. 11d. Find (i) the gain per cent. on the cost price and (ii) what they should have been sold for to make a profit of 15 per cent. on the cost price.*

(i) Let the C.P. be represented by 100 and the S.P. by x ; then by ratios,

$$\begin{aligned}\frac{x}{100} &= \frac{\text{S.P.}}{\text{C.P.}} = \frac{\text{£3 17s. 11d.}}{\text{£3 13s. 4d.}} = \frac{77\frac{11}{12}}{73\frac{1}{3}} \\ &= \frac{935 \times 3}{12 \times 220} = \frac{17}{16}; \\ \therefore x &= \frac{17 \times 100}{16} = \frac{17 \times 25}{4} = \frac{425}{4} = 106\frac{1}{4}.\end{aligned}$$

Hence, when the C.P. is 100, the S.P. is $106\frac{1}{4}$, i.e. the gain is $6\frac{1}{4}$ on 100, which is $6\frac{1}{4}\%$.

\therefore gain on the C.P. = $6\frac{1}{4}\%$.

(ii) If the C.P. is represented by 100, the S.P. will now be represented by $100 + 15 = 115$, so that, denoting the new S.P. by y shillings,

$$\begin{aligned}\frac{115}{100} &= \frac{\text{S.P.}}{\text{C.P.}} = \frac{y}{73\frac{1}{3}}; \\ \therefore y &= \frac{115 \times 73\frac{1}{3}}{100} = \frac{115 \times 220}{100 \times 3} = \frac{253}{3} = 84\frac{1}{3}.\end{aligned}$$

Hence, the new S.P. = $84\frac{1}{3}$ shillings = £4 4s. 4d.

Otherwise : 15% of £3 13s. 4d. = 11s.

\therefore new S.P. = £3 13s. 4d. + 11s. = £4 4s. 4d.

Ex. 10. A retailer buys 3 cwt. 2 qr. 16 lb. of coffee at £9 13s. 8d. per cwt. and sells all of it at 1s. 11d. per lb. Find his gain per cent. on (i) the cost price, and (ii) the selling price.

(i) First find the C.P.

	£	s.	d.	
	9	13	8	= Cost of 1 cwt.
			3	
2 qr. = $\frac{1}{2}$ cwt.	29	1	0	= „ 3 cwt.
16 lb. = $\frac{1}{7}$ „	4	16	10	= „ 2 qr.
	1	7	8	= „ 16 lb.
	35	5	6	= „ 3 cwt. 2 qr. 16 lb.

To find the S.P., 3 cwt. 2 qr. 16 lb. = (336 + 56 + 16) lb. = 408 lb.

and 408 lb. at 2s. 0d. per lb. = £40 16s. 0d.

408 lb. at 1d. per lb. = £ 1 14s. 0d.

∴ 408 lb. at 1s. 11d. per lb. = £39 2s. 0d.

Hence, S.P. = £39 2s. 0d.

C.P. = £35 5s. 6d.

∴ actual gain = £ 3 16s. 6d. = £3 $\frac{33}{40}$.

Thus (i) £3 $\frac{33}{40}$ is gained on £35 5s. 6d. or £35 $\frac{11}{40}$,

so that the percentage gain = $\frac{3\frac{33}{40} \times 100}{35\frac{11}{40}} = \frac{15300}{1411} = 10.8$;

∴ gain on C.P. = 10.8%.

There is no need to find the actual gain in money if the method of Ex. 9 is used; thus, let the C.P. be represented by 100 and the S.P. by x ; then

$$\frac{x}{100} = \frac{\text{S.P.}}{\text{C.P.}} = \frac{39\frac{1}{10}}{35\frac{11}{40}} = \frac{1564}{1411},$$

from which $x = \frac{156400}{1411} = 110.8$;

∴ gain per cent. = 110.8 - 100 = 10.8.

(ii) Gain on S.P. of $\pounds 39\frac{1}{10} = \pounds 3\frac{33}{40}$.

$$\begin{aligned}\therefore \text{percentage gain} &= \frac{3\frac{33}{40} \times 100}{39\frac{1}{10}} = \frac{153 \times 100 \times 10}{40 \times 391} \\ &= \frac{3825}{391} = 9\cdot8.\end{aligned}$$

\therefore Gain on S.P. = $9\cdot8\%$.

The two following examples are of a slightly more difficult type. The methods used should be carefully studied.

Ex. 11. *A tradesman sells his goods at a price 40 per cent. above what they cost him. His overhead expenses are £494 per annum. What must be the average value of goods sold weekly so that 15 per cent. of his takings may be net profit?*

Let the C.P. of the goods bought be represented by £100, then the S.P. is £140 and his gross profit £40.

$$\text{Net profit} = 15\% \text{ of } \pounds 140 = \pounds \frac{15 \times 140}{100} = \pounds 21;$$

\therefore of the gross profit of £40, £21 is net profit, so that the overhead expenses = $\pounds(40 - 21) = \pounds 19$.

Hence, if his weekly takings are £ x , then goods to the value of £ $(52 \times x)$ must be sold annually, and by equal ratios :

$$\frac{52 \times x}{140} = \frac{494}{19},$$

i.e.

$$19 \times 52 \times x = 140 \times 494,$$

from which

$$x = \frac{140 \times 494}{19 \times 52} = 70.$$

\therefore His weekly takings must be £70.

To those familiar with algebra, the following solution will be interesting :

Annual takings = S.P. = $\pounds 52x = \frac{7}{5}$ C.P., giving C.P. = $\frac{5}{7} \times 52x$.

\therefore Gain = $52x - (\text{C.P.} + 494) = 15\% \text{ of } 52x = \frac{3\cdot9}{5}x$;

Hence, $52x - \frac{5}{7} \times 52x - \frac{3\cdot9}{5}x = 494$, from which $x = 70$.

Ex. 12. *A wholesale firm fixes its list prices at 44 per cent. above cost price and, to retail customers, it allows a trade discount of $12\frac{1}{2}$ per cent. Later on, cost prices are increased by ten per cent. but the firm decides to keep the list prices as before and to reduce the trade discount so that the same percentage profit on the cost price is made. Find the new percentage trade discount allowed.*

Suppose the C.P. = £100,

then

List Price = £144,

and the price to the retailer = £144 - $12\frac{1}{2}\%$ of £144

$$= £144 - \frac{1}{8} \text{ of } £144$$

$$= \frac{7}{8} \text{ of } £144 = £126.$$

Hence, the wholesale firm's profit = £126 - £100 = £26, i.e. £26 in a C.P. of £100 or 26%.

Now the increased C.P. = £100 + 10% of £100 = £110, and, since the wholesaler's profit is still to be 26% of the C.P., this profit

$$= £ \frac{26 \times 110}{100} = £28·6.$$

\therefore price to the retailer must be £110 + £28·6 = £138·6 ;

Hence, the discount allowed = £144 - £138·6 = £5·4,

$$\text{so that the percentage discount} = \frac{5·4 \times 100}{144} = \frac{15}{4} = 3\frac{3}{4} ;$$

\therefore new trade discount = $3\frac{3}{4}\%$.

5·7. Average.

When a series of values are known, such for instance as the profits made by a firm over a number of years, comparison is sometimes convenient by imagining the series replaced by another consisting of equal values of the same total sum. Each of these equal values is known as the **Average** or **Mean Value** of the first series. It is obvious that this average value may readily be obtained by dividing the sum of the given values by their number.

Ex. 13. *The revenue of Great Britain for the years 1932-3, 1933-4, 1934-5, 1935-6, 1936-7 was, in millions of pounds : £744·791, £724·567, £716·441, £752·920, £797·289. Calculate the average revenue for these five years.*

	£
Revenue in millions for 1932-3 =	744·791
„ „ „ 1933-4 =	724·567
„ „ „ 1934-5 =	716·441
„ „ „ 1935-6 =	752·920
„ „ „ 1936-7 =	797·289
Hence, total for 5 years =	<u>3736·008</u>

∴ Average = one-fifth of this total = 747·2016

the average revenue = 747·2016 millions

= £747,201,600.

The average to the nearest thousand would be £747,202,000.

Ex. 14. *The profits of a business undertaking for six consecutive years were £19,462, £21,314, £22,016, £16,571, £18,462, £20,597.*

Calculate the percentage that (i) the best year's profit exceeded the average, and (ii) the worst year's profit was below the average.

Give the results correct to two places of decimals.

Here the average profit must first be found ; hence, adding the given profits and dividing by 6 :

£		£
19,462	(i) Now the best year's profit =	22,016
21,314	Average profit	= <u>19,737</u>
22,016	Excess over average	= <u>2,279</u>
16,571	∴ Percentage of average	
18,462		$= \frac{2279 \times 100}{19737} = \frac{227900}{19737} = 11·545...$
20,597		
6) <u>118,422</u>	Hence, the best year's profit exceeded the	
19,737	average by 11·55%.	

(ii)

Average profit = £19,737

Worst year's profit = £16,571

Defect from average = £3,166

$$\therefore \text{Percentage of average} = \frac{3166 \times 100}{19737} = \frac{316600}{19737} = 16.04.$$

Hence, the worst year's profit was below the average by 16.04%.

EXERCISES 5

A. Ratio and Proportion

Find, in its simplest form, the ratio of :

1. £4 19s. 9d. to £6 1s. 11d.
2. 3 qr. 27 lb. to 2 cwt. 1 qr. 7 lb.
3. 33 rupees to £2 11s. 4d., when the rupee is worth 1s. 5½d.
4. 14 zlotys to 19 kroner, when £1 = 19.95 kroner = 24.5 zlotys.
5. The weight of gold in three bars, each $\frac{1}{12}$ ths fine, to that of five bars of the same size but of 18-carat gold.
6. 12.5 oz. Troy to 1.5 lb. Avoirdupois.
7. Find the ratio of 10.5 dollars to £2 16s., when the rate of exchange is 4.75 dollars to £1.
8. A man received £12 6s. 10½d. for a job which took 79 hours to complete. On another job he received £9 15s. 10d. for 47 hours' work. Express, in its simplest form, the first rate of pay as a ratio of the second.
9. A streamline train travels 76 miles in one hour and a ship sails at 27.5 knots. Taking a knot as a speed of 6080 feet per hour, express the average speed of the train as a ratio, in its simplest form, of that of the ship.
10. If 4 cwt. 3 qr. 27 lb. of material has to be shared among two customers in the ratio of 5 : 8, what weight should each have?
11. If 5¼ yards of material cost 13s. 6¾d., find the cost of 9¾ yards.
12. If 8 tons 16 cwt. of coal cost £13 4s., how much must be paid for a twelve-ton truck load?
13. A roll of cloth contains 16¾ yards and costs £1 19s. 1d.

How much more would a similar roll containing $22\frac{1}{2}$ yards cost at the same price per yard?

14. Divide £3 10s. 6d. in the ratio of 8 : 11 : 17.

15. An alloy contains three metals, A , B , C . One-fourth of the total weight is due to A and one-fifth of the remaining weight is B .

(i) What is the ratio of the weight of A to that of C , and

(ii) What must be the weight of a piece of the alloy which contains 5·4 lb. of B ?

16. A brass alloy consists of 160 parts by weight of copper, 25 parts of tin and 5 parts of zinc. Taking the price per ton of copper, tin and zinc as £45, £235 and £18 8s. respectively, find the value of the metal in a brass casting weighing 360 lb. (U.L.C.I.)

17. The half of a legacy has to be divided among P and Q in the ratio 4 : 3, whilst the other half has to be divided between them in the ratio of 2 : 5. Find the ratio of the total amount that P and Q each receive.

18. Two equal quantities of a liquid are diluted with water so that the ratio of the water to the liquid in the first is 2 : 3, and in the second, 8 : 15. The liquids are poured together into a larger vessel and thoroughly mixed. Find the ratio of the water to the liquid in this mixture.

19. The capital of a company is £473,200, and at the end of the year a net profit of £44,863 is made. This is used by transferring part to the reserve fund, carrying forward another part to next year and paying out the rest to the shareholders as dividend. The division into these three parts is made in the ratio of 7 : 9 : 13. Find (i) the dividend, and (ii) the ratio of the dividend to the capital.

20. Three men, A , B , C , form a partnership, their respective capitals being £1551, £2368, £4221. The profits are to be divided in proportion to capital. At the end of the year, it was found that C 's share of the profit exceeded B 's share by £254 15s. 9d. Find (i) the total profit, and (ii) the respective shares of A , B and C .

(U.L.C.I.)

B. Statistical Percentages

21. Up to the end of August, 1938, the number of wireless licences issued by the Post Office was 8,689,850, whilst for the corresponding period in 1937 the number was 8,305,950. Calculate, to two places of decimals, the percentage increase.

22. The takings of a railway company fell from £4,735,287 16s. 5d. to £4,112,236 11s. 8d. Find the percentage decrease correct to one place of decimals. (L.Ch.C.)

23. On March 31st, 1934, the aggregate National Debt consisted of the following :

External Debt	-	-	£1,036,545,184.
Internal Debt	-	-	£6,908,649,225.
Other Liabilities	-	-	£208,064,057.

Express the Internal Debt as a percentage of the whole, correct to two places of decimals. (L.Ch.C.)

24. From the following table, calculate the exact percentage of reduction made in each case.

Ordinary Price	-	-	£1 8s. 8d.	13s. 4d.	£2 12s. 1d.
Sale Price	-	-	17s. 11d.	7s. 8d.	£2 8s. 7d.

25. The following table gives the sums paid out in 1934 and 1935 under the Workmen's Compensation Act.

Paid out in			1934	1935
Fatal Cases	-	-	£656,765	£770,118
Non-Fatal Cases	-	-	£4,618,866	£4,940,009

Calculate, correct to three significant figures, the percentage increase in the total paid in 1935 over the total paid in 1934.

26. The following table gives particulars of the passenger traffic on a certain railway for the years 1934, 1935 :

	1934		1935	
	Number	Receipts	Number	Receipts
		£		£
Ordinary Passengers -	156,961,480	9,100,625	159,997,665	9,376,529
Workmen -	65,102,992	979,009	68,021,996	1,038,463
Season Tickets -	193,129	2,837,586	202,035	2,972,495

Calculate, in each case to one place of decimals, the following percentage increases which the figures of 1935 show on the corresponding figures for 1934 :

- (i) in the total receipts,
 (ii) in the number of passengers, excluding season ticket holders,
 (iii) in the receipts from season ticket holders. (R.S.A.)

27. In 1931, of 19,133,010 men in England and Wales, 8,489,813 were married, 732,402 were widowers and the remainder were single. Calculate, to three significant figures, the percentage of the total number of men who had not been married.

28. The net results from income tax are shewn below for two years :

1934-35			1935-36
England	-	-	£215,362,010
Scotland	-	-	£12,153,763
			£223,024,978
			£12,629,651

Calculate the percentage increase, to two places of decimals, in the total receipts for 1935-36 over the total for 1934-35.

29. The profits of a company in the year 1920 were 64 per cent. higher than in 1919 ; in 1921 they were 54 per cent. lower than in 1920 ; in 1922 they were 49 per cent. lower than in 1921 ; in 1923 they were 112 per cent. higher than in 1922. By what percentage were the profits in 1923 higher or lower than those in 1919?

(R.S.A.)

30. In the year 1934 the working expenses of a railway were 85.1 per cent. of the gross takings. In 1935 the gross takings increased by 9 per cent. and the working expenses were 84.7 per cent. of those takings. Find by what percentage, to the nearest whole number, the net profit in 1935 was greater than that in 1934.

(R.S.A.)

C. Commercial Percentage

31. A house agent's charge for letting property is $7\frac{1}{2}$ per cent. on each year's rental. What will the agent get per year for letting a house at an annual rent of £32?

32. An article sold for £3 17s. 7d. realised a profit of $22\frac{1}{2}\%$ on the cost price. Find the cost price.

33. A fruiterer bought 2 cwt. of apples for £2 7s. 3d. and found that 8 lb. were unsaleable. At what price per lb. must he sell the remainder in order to make a profit of 25% on the selling price?

34. A retail clothier marks his goods so that 35 per cent. of the marked price is profit. Find (i) the cost price of a suit which he marks at £6 5s. ; (ii) at what price he will mark an overcoat which cost him £2 18s. 6d. (U.L.C.I.)

35. A dealer paid a car manufacturer £285 for a car. What should be his selling price for the car if, after allowing a buyer 5% discount on the selling price, he made a profit of 25% on the amount he received? (L.Ch.C.)

36. An electricity company makes a fixed annual charge of 9% of the rateable value of a house, chargeable quarterly, and then an additional charge of $\frac{3}{4}$ d. per unit of electricity consumed. A householder whose house has a rateable value of £35 has an electricity account for a period of three months amounting to £1 10s. 6d. How many units of electricity were consumed?

37. A grocer bought a bag of sugar containing 148 lb. for £1 17s. On weighing into one-pound packets there was a loss of 4 lb. He sold 48 lb. at $3\frac{1}{2}$ d. per lb. and had then to reduce the price. What was the new price per lb. if he made a profit of $7\frac{1}{2}$ % on the selling price of the whole bag? (L.Ch.C.)

38. A draper sells some cloth at 9s. 9d. per yard and gains 17% on the cost price. Later on, he has to reduce the price to 8s. $10\frac{1}{2}$ d. per yard. What is his percentage profit on the cost price now?

39. A trader adds 28% to the cost price of goods to obtain the selling price. During the year 1936 the sales amounted to £3896 16s. Find his gross profit for the year. (U.L.C.I.)

40. A tradesman sells his goods at a price 30% above what they cost him. If his overhead expenses are £578 per annum, what must be the average value of goods sold weekly so that 10% of his takings may be net profit? (R.S.A.)

41. A wholesale dealer sold blankets at 22s. 6d. per pair, less a trade discount of $12\frac{1}{2}$ % and a further cash discount of $2\frac{1}{2}$ %. What was the net amount paid by a customer who bought 560 pairs? (U.L.C.I.)

42. A decorator's quotation for a job was £10 15s. for labour and £6 5s. for material. Before the work is done, however, labour charges are raised by $2\frac{1}{2}$ % and the cost of materials decreased by $4\frac{1}{2}$ %. What will the work now cost, and what difference will there be between the actual charge and that given in the quotation?

43. A commercial traveller sold 4248 lb. of tea at 2s. 1d. per lb. and 1125 lb. of coffee at 2s. 4d. per lb. His total commission on these sales amounted to £15. If, on his sales for tea, he was paid $2\frac{1}{2}\%$ commission, what percentage was he paid on sales of coffee?
(U.L.C.I.)

44. A middleman bought goods for £850. He sold them for £1150 and his expenses in connection with their sale amounted to £175. Express (i) his net profit as a percentage of his cost price ; (ii) his expenses as a percentage of his sales, giving each result correct to three significant figures.
(U.L.C.I.)

45. An agent received as commission $1\frac{3}{4}\%$ of the value of his purchases and $3\frac{1}{4}\%$ of the value of his sales. His purchases during a certain period amounted to £4321 and his sales to £5678. Find, to the nearest penny, his total commission for the period and the average percentage, correct to two decimal places, he received.
(U.L.C.I.)

46. A man bought a horse for a certain sum and sold it again at a loss of ten per cent. on his outlay. If he had received £9 more he would have gained $12\frac{1}{2}\%$ on his outlay. What did the horse cost him?
(B.M.I.)

47. The real cost of an article is 57% of the price at which it is marked for sale. It is sold at a discount of 5% for cash. What percentage of profit on the actual selling price does the dealer make?

48. *A* and *B* formed a partnership, the total capital being £6640, of which £2407 was provided by *A* and the rest by *B*. It was arranged that they should divide equally annual profits up to 15% of the total capital, but that profit in excess of 15% should be divided proportionally to their shares of the capital. The first year's profit was £1567 ; how should this be shared between *A* and *B*?
(U.L.C.I.)

49. The selling price of an article is fixed by adding 30% to the cost of materials and labour. If the costs of materials and labour are in the ratio of 5 : 7 and the materials for one article cost 15s., find the selling price of 50 articles.
(U.L.C.I.)

50. A retailer, buying goods from the manufacturer, is allowed a trade discount of 25% from the latter's price list. The former sells the goods to customers at a price equal to the list price, but

allows a rebate of one shilling in the £ for ready money. What percentage is the retailer's profit of the amount he paid for the goods? (R.S.A.)

51. A radio manufacturer stops making a set which he lists at £12. Usually he allows the retailer 30% discount off the list price and still makes a profit of 25% on the price received. In order to clear his stock of this set, he offers them to the retailer at 22s. below cost price. What profit per cent. on the selling price does the retailer make if he sells each set for £6 17s. 6d.?

52. A draper buys gloves at £5 a dozen pairs and is allowed 5% discount. In a week he sells six dozen pairs at 10s. 6d. a pair. The manufacturer increases the price to the draper, allowing the same percentage discount and, in consequence, the draper increases the price to 11s. 6d. a pair. His sales drop and, though his weekly receipts increase by three shillings, his profit decreases by 1s. 9d. What was the manufacturer's percentage increase in price?

(L.Ch.C.)

53. A book is sold to a trade bookseller at $33\frac{1}{3}\%$ less than the published price, and the bookseller's sale price is 8% higher than the trade price. What percentage of the published price is the bookseller's sale price, and what is the published price of a book which he sells at 13s. 6d.?

54. A retailer bought typewriters from a manufacturer at a catalogue price of £12 10s. each, but was allowed a 20% trade discount. He sold a typewriter at the catalogue price, but took in part payment an old machine on which he allowed £3, and further allowed 5% discount for cash on the resulting bill. He had the old machine reconditioned by a workman in six hours whom he paid at the rate of 1s. 10d. per hour, and then sold the machine for £5 5s. Find the percentage profit on his total outlay. (L.Ch.C.)

55. A wholesaler fixes his list prices at 44% above cost price and to his retail customers he allows a trade discount of $12\frac{1}{2}\%$. When the cost prices rise by 9%, the wholesaler keeps his list prices unaltered but, in order to make the same profit as before, reduces the trade discount. Find the new percentage of trade discount allowed.

56. Milk contains by bulk 88.75 per cent. water, 2.75 per cent. of fat and 8.5 per cent. of non-fatty solids. A milkman added water to the milk so that, on analysis, the percentage of fat was

only 2·2. Find (i) the number of gallons of water added to 100 gallons of milk; (ii) the amount by which a person was defrauded who bought ten gallons of milk at 2s. 1d. a gallon.

57. By selling an article for $16\frac{1}{2}$ guineas a dealer reckons that he will make $15\frac{1}{2}\%$ profit on his outlay. What percentage profit does he make on his outlay if the selling price is cut down to £16 10s.? (R.S.A.)

58. A tin of polish containing 8 ounces of polish is sold for 9d. The polish costs 10d. a lb. to make and the tins 18s. a gross. When the cost of the polish increases by 12%, half an ounce less of polish is put into each tin. Find the percentage profit now made on the cost price if the selling price is unaltered. (L.Ch.C.)

59. A manufacturer fixes his list prices at 25% above cost price and he allows his retail customers a trade discount of 15%. The retailer sells at the manufacturer's list price on the following terms: 12% cash down and the remainder, increased by $2\frac{1}{2}\%$ of its value, by regular instalments. What percentage profit, to three significant figures, on his outlay does the retailer make when all instalments are paid?

60. A retailer buys goods from a wholesaler at a price which is 25% less than the latter's list price. He sells them at a price 5% less than this list price. What is the retailer's percentage profit based on (i) his outlay, (ii) his sales? (R.S.A.)

61. A boot dealer sold 80 pairs of a certain boot in three months at 17s. 6d. a pair, making 30% profit on the selling price. In the hope of increasing his sales, he reduced the price by 1s. 6d. a pair, with the result that his total profit for three months decreased by 15s. Find the percentage increase in the number of pairs sold. (L.Ch.C.)

62. Goods are sold in France at 31·5 francs per kilogram, and the duty of $33\frac{1}{3}\%$ of their cost in France must be paid when they are shipped to England. Find the selling price per lb. in England, correct to the nearest penny, if 10% profit is made on the outlay. Take £1 to be equivalent to 175 francs and 1 kilogram to 2·2 lb.

63. A barrel of butter containing 100 kilograms is bought for 198 kronen in Denmark, including cost of carriage to England. It is there subject to an import duty of 15s. per cwt. Find the price per lb. in England at which it must be sold to make a profit of 20% on the outlay, taking 1 kilogram = 2·2 lb. and £1 = 22·4 kronen.

D. Averages

64. The attendance at an annual exhibition for five consecutive years was : 123,593 ; 140,627 ; 161,128 ; 198,070 ; 180,752. Find (i) the average yearly attendance ; (ii) what the attendance must be in the sixth year in order to raise the average to 175,000.

65. The receipts from income tax during the years shewn are given in the following table :

Years	Receipts
1924-25	£251,766,736
1925-26	£237,204,982
1926-27	£210,954,229
1927-28	£233,790,790
1928-29	£220,086,381
1929-30	£218,851,564
1930-31	£235,553,636

Find the total receipts for the seven years. If the average for 8 years, including 1931-32, is £234,465,172, find the receipts for the year 1931-32. (L.Ch.C.)

66. A man buys the following £1 shares :

58 at 28s. 4d. ; 37 at 25s. 6d. ; 46 at 29s. 9d.

Find the average price paid per share.

67. In the course of a week, a grocer sells $40\frac{1}{4}$ lb. of tea at 2s. 6d. per lb., $25\frac{1}{2}$ lb. at 2s. 9d. per lb. and $13\frac{3}{4}$ lb. at 3s. 2d. per lb. Find, to the nearest farthing, the average price of all the tea sold. (R.S.A.)

68. A householder consumed 12,400, 9,400, 7,600 and 12,200 cubic feet of gas in the four quarters of a certain year. Taking 1000 cubic feet to be equivalent to $5\frac{1}{2}$ therms, find the average weekly cost of the gas used when the price of one therm is $8\frac{3}{4}$ d. Take 52 weeks to the year.

69. A tea blender mixes together all the tea contained in a chest costing 27s. 6d. ; 12 lb. from another chest costing 37s. 6d. the chest ; and 8 lb. from a third chest which costs 45s. the chest. He sells the mixture at a uniform price which gives him a profit of 15% on his sales. Find the price at which he sells it per lb., taking a chest to hold 20 lb. of tea.

70. The sales of three classes of goods during a given period were: £5175, £7475, £4025, and the profits made on these sales were 12%, 28% and 32% respectively. Calculate the average percentage of profit on the total sales.

71. The profits of a company for the last five years have been as follows:

Year	-	1933	1934	1935	1936	1937
Profits	-	£10,957	£13,752	£17,040	£17,639	£20,961

Find the percentage increase, correct to two decimal places, for the year 1937:

- (i) on the profits for the year 1933,
 - (ii) on the annual average profits for the four years 1933-1936.
- (U.L.C.I.)

72. The profits of a business for seven consecutive years were: £63,824; £68,753; £83,419; £71,257; £37,596; £12,469; £29,573. Find by what percentage (i) the best year's profit exceeded the average, (ii) the worst year's profit fell short of the average.

(U.L.C.I.)

73. A tradesman's takings for seven consecutive weeks were as follows: £24 6s. 9d.; £21 11s. 10d.; £18 18s. 9d.; £23 19s. 2d.; £25 6s. 3d.; £23 17s. 1d.; £26 1s. 5d. Find by what percentage the lowest weekly takings were below the average.

74. To a class of 16 boys are added 6 new boys whose average age is eleven months less than the average age of the 16. By how many months is the average age of the whole class lowered?

(R.S.A.)

75. A man buys the following £100 shares:

No. of shares	Price each	Annual dividend on each
52	£98 10s.	£3 10s.
81	£94 5s.	£2 15s.
29	£99 15s.	£4 5s.
17	£103	£5 15s.

Calculate to the nearest penny,

- (i) the average price paid per share,
- (ii) the average annual dividend paid on each share.

76. The average age of a class of 17 boys at the end of a term was 14 years 3 months. Of these five left of the average age of 15 years 2 months and, at the beginning of next term two months later, six new boys of average age 13 years 11 months came into the class. What was then, to the nearest month, the average age of the class? (R.S.A.)

77. The profit made by a firm in 1932 was 15% higher than that made in 1931; in 1933 it was 10% less than in 1932; in 1934 it was 8% higher than in 1933, and in 1935 it was 4% less than in 1934. If the profit in 1931 was £46,875, calculate the average annual profit for the five years 1931-35.

78. A commercial traveller visited three towns. At the first he stayed 44 days at an average cost of 27s. 6½d. per day, and at the second he stayed 59 days at an average cost of 28s. 5d. per day. At the third town his stay cost him £56 11s. and the average daily cost of the whole tour was 28s. 11d. How many days did he stay at the third town and what did it cost him per day?

79. A firm bought from the government a quantity of surplus material at 40 per cent. below the cost price of manufacture. One-eighth of the stock was unsaleable, three-quarters of it were resold at 30 per cent. profit, and the remainder at 20 per cent. profit. Find

- (i) the firm's average profit per cent. on the transaction,
- (ii) the cost of manufacture to the government, to the nearest £, if the firm's gross receipts were £62,000. (C.I.S.)

80. The average profit per annum made by a business undertaking for three years was £4325; for the first and second years the average yearly profit was £4268 10s., whilst the average for the second and third years was £4295. Find

- (i) the profit for each year,
- (ii) the percentage below the average of the lowest year's profit.

CHAPTER VI

SIMPLE INTEREST

6.1. Definitions.

WHEN the use of an article or a building is required temporarily, payment must be made for the loan or hire. Similarly, when money is borrowed, payment is demanded for the use of it. Such payment is known as **Interest**, whilst the sum borrowed is called the **Principal**. The same practice is applied to investment; when money is deposited in a Savings Bank, for instance, interest is paid to the depositor, as the money is considered a loan to the bank for the time being.

The sum of the Interest and the Principal is called the **Amount**.

Interest is reckoned as a percentage of the Principal, and is generally charged at the end of each year the borrowed money is held; it is expressed as the rate per cent. per annum. For a fraction of a year the interest is calculated as that fraction of the interest chargeable for the whole year.

When, for a fixed sum borrowed at a given rate, the interest charged each year is the same, the sum is said to be borrowed at **Simple Interest**; this is briefly indicated by the letters **S.I.**

Ex. 1. Find the simple interest on £584 at $3\frac{1}{2}$ per cent. per annum (i) for 5 years, (ii) for the period June 9th to December 31st, 1938.

It is often convenient to find the S.I. on £1 for 1 year at the given rate. In this question,

S.I. on £100 for 1 year at $3\frac{1}{2}\%$ per annum = £3.5,

\therefore S.I. on £1 for 1 year at $3\frac{1}{2}\%$ per annum = £0.035.

(i) Hence, S.I. on £584 for 5 years at $3\frac{1}{2}\%$ per annum

$$= £(0.035 \times 584 \times 5) = £102.2 = \text{£}102 \text{ 4s.}$$

(ii) From June 9th to December 31st, the number of days

$$= 21 + 31 + 31 + 30 + 31 + 30 + 31 = 205.$$

Now $205 \text{ days} = \frac{205}{365} \text{ year} = \frac{41}{73} \text{ year}.$

$$\begin{aligned} \therefore \text{S.I. on } £584 \text{ for } \frac{41}{73} \text{ year at } 3\frac{1}{2}\% \text{ per annum} \\ = £(0.035 \times 584 \times \frac{41}{73}) = £(0.035 \times 8 \times 41) = £11.48 \\ = £11 \text{ 9s. 7d. to the nearest penny.} \end{aligned}$$

6.2. The General Formula.

From the definitions given in the last Section, a general rule or formula applicable to the calculations arising in simple interest problems may readily be deduced.

Let a sum of $£P$ be put out at simple interest for n years at r per cent. per annum; then the

$$\text{S.I. on } £100 \text{ for 1 year at } r\% \text{ per annum} = £r,$$

$$\therefore \text{S.I. on } £1 \text{ for 1 year at } r\% \text{ per annum} = £\frac{r}{100}.$$

$$\text{Hence, the S.I. on } £P \text{ for } n \text{ years at } r\% \text{ per annum} = £\frac{P \times n \times r}{100}.$$

i.e. the simple interest is the product of the Principal, rate and time in years, divided by 100.

Again, if $£P$ put out at simple interest for n years at r per cent. per annum amounts to $£A$, then

$$A = P + \text{S.I.} = P + \frac{P \times n \times r}{100} = P \left(1 + \frac{n \times r}{100} \right).$$

Hence the simple interest and amount are given by the following important relations :

$$\left. \begin{aligned} \text{(i) S.I.} &= \frac{P \times r \times n}{100} \\ \text{(ii) } A &= P \left(1 + \frac{r \times n}{100} \right) \end{aligned} \right\} \dots\dots\dots (1)$$

The formula (i) may also be used to find any one of the quan-

tities involved when the other three are given. Thus, multiplying (i) throughout by 100,

$$100 \times \text{S.I.} = P \times n \times r,$$

and from this it is easily seen that

$$(i) P = \frac{100 \times \text{S.I.}}{n \times r}; \quad (ii) n = \frac{100 \times \text{S.I.}}{P \times r}; \quad (iii) r = \frac{100 \times \text{S.I.}}{P \times n} \dots\dots(2)$$

The following simple problems will shew how the above formulae may be applied.

Ex. 2. *The simple interest on a sum of money for $7\frac{1}{2}$ years at $2\frac{1}{2}$ per cent. per annum is £139 10s.; find the sum.*

Here $\text{S.I.} = £139\frac{1}{2}$; $n = 7\frac{1}{2}$; $r = 2\frac{1}{2}$.

$$\therefore \text{from 2 (i), the Principal } P = \frac{100 \times 139\frac{1}{2}}{7\frac{1}{2} \times 2\frac{1}{2}} = \frac{100 \times 279 \times 2 \times 2}{2 \times 15 \times 5}.$$

$$= 4 \times 93 \times 2 = 744.$$

Hence, the required sum of money is £744.

Ex. 3. *Find the rate per cent. per annum when £568 amounts to £738 8s. in eight years at simple interest.*

In this case, $A = £738 \text{ 8s.} = £738\frac{2}{5}$; $P = £568$ and $n = 8$.

Hence, the $\text{S.I.} = A - P = £(738\frac{2}{5} - 568) = 170\frac{2}{5}$, so that, from 2 (iii), the rate per cent. per annum

$$= \frac{100 \times 170\frac{2}{5}}{568 \times 8} = \frac{100 \times 852}{568 \times 8 \times 5} = \frac{15}{4} = 3\frac{3}{4},$$

i.e. the required rate is $3\frac{3}{4}\%$ per annum.

By using 2 (ii), the time in years may be found in precisely the same way.

6.3. A Special Rule when Time is given in Days.

In the above simple exercises, the data are purposely chosen so that the resulting calculations may work out exactly. This is very rarely the case in practice, and many devices have been brought

into use to simplify and shorten the arithmetical work. Frequently the time is not an exact number of years, but is given in days. This renders the calculation of simple interest even more cumbersome and, as a consequence, an approximate rule is often used. This rule will now be considered briefly.

Let d denote the number of days, then taking 365 days to the normal year, $n = d/365$;

$$\therefore \text{S.I.} = \frac{P \times r \times d}{100 \times 365}.$$

Now, if the denominator could be transformed approximately into a multiple of ten, the calculation would be considerably simplified.

Let k be any number, then for all values of k ,

$$\text{S.I.} = \frac{P \times r \times d \times k}{100 \times 365 \times k}.$$

So far k may be any number whatsoever; suppose it be chosen so that $365 \times k = 1000$,

$$\text{then } k = \frac{1000}{365} = \frac{200}{73} = 2 \times \frac{100}{73} = 2 \times 1.369863 \dots$$

$$= 2 \times 1.3699, \text{ correct to four places.}$$

$$\begin{aligned} \text{But } 0.369999 \dots &= (0.333333 \dots) + (0.033333 \dots) + (0.003333 \dots) \\ &= \frac{1}{3} + \frac{1}{30} + \frac{1}{300}. \end{aligned}$$

Hence, as a near approximation, k may be taken as

$$2 \times \left(1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}\right).$$

Substituting this for k in the above formula,

$$\text{S.I.} = \frac{P \times r \times d \times 2 \times \left(1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}\right)}{100,000} \dots \dots \dots (3)$$

To find the error in using this formula, the simplified value of k or $1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}$ is $\frac{137}{100}$.

Hence, the error is

$$\begin{aligned} \frac{P \times r \times d \times 2 \times 137}{10,000,000} - \frac{P \times r \times d}{100 \times 365} &= \frac{P \times r \times d}{100} \times \left\{ \frac{274}{100,000} - \frac{1}{365} \right\} \\ &= \frac{P \times r \times d}{100} \times \frac{1}{10,000 \times 365} = \frac{P \times r \times d}{100 \times 365} \times \frac{1}{10,000}. \end{aligned}$$

Thus, the actual error in the use of the approximate formula is that it gives a result too large by 1 in 10,000. This is 0·24 pence in £10, or 0·96 pence in £40, so that, in calculating the simple interest to the nearest penny, one penny must be deducted for each £40 of the interest.

6·4. Another Method for Time given in Days.

The rule of Section 6·3 is not popular; many prefer to use the Continental method of reckoning 360 days to the year and afterwards correcting the value thus obtained.

From the formula of Sect. 6·3 :

$$\text{S.I.} = \frac{P \times r \times d}{100 \times 365} = \frac{P \times r \times d \times 360}{100 \times 365 \times 360}.$$

$$\text{But} \quad \frac{360}{365} = \frac{365 - 5}{365} = 1 - \frac{5}{365} = 1 - \frac{1}{73};$$

$$\therefore \text{S.I.} = \frac{P \times r \times d}{100 \times 360} \times \left(1 - \frac{1}{73} \right). \dots\dots\dots(4)$$

Hence, when 360 is used in the denominator instead of 365, it is necessary to subtract $\frac{1}{73}$ rd of the value thus obtained.

The following calculation will illustrate the application of each rule.

Ex. 4. Calculate, to the nearest penny, the simple interest on £2973 for 128 days at $4\frac{1}{2}$ per cent. per annum.

By the approximate formula (3), the simple interest is

$$\begin{aligned} &\pounds \frac{2973 \times 4\frac{1}{2} \times 128 \times 2 \times \left(1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300} \right)}{100,000} \\ &= \pounds \{ 34 \cdot 24896 \times (1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}) \}. \end{aligned}$$

Now £34.24896 $\times 1$ = £34.24896

„ $\times \frac{1}{3}$ = £11.41632

„ $\times \frac{1}{30}$ = £1.14163 retaining only five places.

„ $\times \frac{1}{300}$ = £0.11416 „ „ „ „

\therefore S.I. = £46.92107 = £46 18s. 5d.

Since this sum is greater than £40 and less than £80, it is too large by one penny; hence the required interest is £46 18s. 4d.

By the Continental method given in (4),

$$\begin{aligned}\text{S.I.} &= \text{£} \frac{2973 \times 9 \times 128}{100 \times 360 \times 2} \times \left(1 - \frac{1}{73}\right) \\ &= \text{£} 47.568 \times \left(1 - \frac{1}{73}\right) = \text{£}(47.568 - 0.6516) \\ &= \text{£} 46.9164 = \text{£} 46 \text{ 18s. 4d. to the nearest penny.}\end{aligned}$$

To check this calculation, the ordinary method gives, as the S.I.

$$\begin{aligned}\text{£} \frac{2973 \times 4\frac{1}{2} \times 128}{100 \times 365} &= \text{£} \frac{2973 \times 9 \times 128}{1000 \times 73} = \text{£} \frac{3424.896}{73} \\ &= \text{£} 46 \text{ 18s. 4d. to the nearest penny.}\end{aligned}$$

The approximate formula (3) really only lightens the calculation in respect of the division; the long division by 73 or 365 is replaced by a short division by 3, but the long multiplication remains the same. In (4) the evaluation of the fraction is easier with 360 in the denominator instead of 365, but the division by 73 is still necessary. It is therefore for the person who has to make a calculation to decide from the data which method will prove to be the more convenient.

6.5. A S.I. Table of Nine Multiples.

Where the calculations are numerous, it is sometimes convenient to construct first a table of nine multiples giving the simple interest on £1 for 1 day at 1 per cent. per annum.

Now the S.I. on £1 for d days at 1% per annum = $\text{£} \frac{1 \times 1 \times d}{100 \times 365}$; but this might also represent the S.I. (i) on £1 for 1 day at $d\%$

per annum, or (ii) on £ d for 1 day at 1% per annum. Hence, the use of the following table.

d	Simple Interest on (i) £1 for d days at 1% per annum, or (ii) £1 for 1 day at $d\%$ per annum, or (iii) £ d for 1 day at 1% per annum.
1	£0.000027397260
2	£0.000054794521
3	£0.000082191781
4	£0.000109589041
5	£0.000136986301
6	£0.000164383562
7	£0.000191780822
8	£0.000219178082
9	£0.000246575342

To apply this table, Ex. 4 will now be re-worked.

Ex. 5. *By using the above table, calculate the simple interest on £2973 for 128 days at $4\frac{1}{2}\%$ per cent. per annum to the nearest penny.*

From the table, the

S.I. on	£1 for 100 days at 1% per ann.	= £0.0027397260
„	„ 20 „ „ „	= £0.0005479452
„	„ 8 „ „ „	= £0.0002191781
∴ „	„ 128 „ „ „	= £0.0035068493
„	„ „ „ 4% „	= £0.0140273972
„	„ „ „ $\frac{1}{2}\%$ „	= £0.0017534247
∴ „	„ „ „ $4\frac{1}{2}\%$ „	= £0.0157808219

Hence,

S.I. on	£2000 for 128 days at $4\frac{1}{2}\%$ per ann.	= £31.5616438
„	£900 „ „ „ „	= £14.2027397
„	£70 „ „ „ „	= £1.1046575
„	£3 „ „ „ „	= £0.0473425
∴ „	£2973 „ „ „ „	= £46.9163835
		= £46.916, to three decimal places = £46 18s. 4d.

Note that, in this case, the Principal £2973 might have been taken as £3000 + £3 - £30. Also, alternatively, the calculation might have commenced by finding the S.I. on £2973 for 1 day at 1% per annum. This is left for the student to try as a check on the result already obtained by two independent methods.

6-6. Miscellaneous Problems.

Some typical problems involving simple interest will now be discussed.

Ex. 6. *A sum of £584 was borrowed at $3\frac{1}{2}$ per cent. per annum on March 9th, 1938, subject to the condition that the debt was to be paid off later in the year by a single payment of £599 8s. Find the date upon which the repayment was due.*

Let the number of days between March 9th and the date of repayment be d , then the

S.I. on £584 for d days at $3\frac{1}{2}$ per cent. per annum

$$= \pounds \frac{584 \times d \times 3\frac{1}{2}}{100 \times 365} = \pounds \frac{7}{125} \times d.$$

But the simple interest = £599 8s. - £584 = £15 8s. = £15 $\frac{2}{5}$,

$$\therefore \frac{7}{125} \times d = 15\frac{2}{5} = \frac{77}{5},$$

so that

$$d = \frac{77}{5} \times \frac{125}{7} = 275.$$

Now from March 9th to December 1st, there are

(22 + 30 + 31 + 30 + 31 + 31 + 30 + 31 + 30 + 1) days = 267 days,

and (275 - 267) days = 8 days.

Hence, the repayment is due on **December 8th, 1938.**

Ex. 7. *A trader borrows £658 from his bank on March 15th ; on the following September 1st he repaid £438 and on December 31st he settled the debt by a payment of £236 3s. What was the rate per cent. per annum of interest charged by the bank ?*

From March 15th to September 1st, there are

$$(16 + 30 + 31 + 30 + 31 + 31 + 1) \text{ days} = 170 \text{ days,}$$

and from September 1st to December 31st, there are

$$(30 + 31 + 30 + 31) \text{ days} = 122 \text{ days.}$$

Now the trader has the use of £658 for 170 days and the remainder, £658 - £438 = £220, for 122 days.

Hence, if the rate per cent. per annum = r , then the

$$\begin{aligned} \text{S.I. on £658 for 170 days at } r\% \text{ per annum} \\ = \text{£} \frac{658 \times 170 \times r}{100 \times 365} = \text{£} \frac{329 \times 17 \times r}{1825} = \text{£} \frac{5593}{1825} \times r, \end{aligned}$$

and the S.I. on £220 for 122 days at $r\%$ per annum

$$= \text{£} \frac{220 \times 122 \times r}{100 \times 365} = \text{£} \frac{11 \times 122 \times r}{1825} = \text{£} \frac{1342}{1825} \times r.$$

$$\begin{aligned} \therefore \text{the total interest} &= \text{£} \left(\frac{5593}{1825} + \frac{1342}{1825} \right) \times r \\ &= \text{£} \frac{6935}{1825} \times r = \text{£} \frac{19}{5} \times r. \end{aligned}$$

But the S.I. charged = £236 3s. - £220 = £16 3s. = £16 $\frac{3}{20}$;

$$\therefore \frac{19}{5} \times r = 16 \frac{3}{20} = \frac{323}{20},$$

so that

$$r = \frac{323 \times 5}{20 \times 19} = \frac{17}{4} = 4\frac{1}{4}.$$

\therefore the required rate of interest = $4\frac{1}{4}\%$ per annum.

This might have been worked by finding, in terms of r , the interest on £658 for the whole period, i.e. (170 + 122) days or 292 days, and then subtracting the interest on £438 for 122 days; this is

$$\text{£} \left(\frac{658 \times 292 \times r}{100 \times 365} - \frac{438 \times 122 \times r}{100 \times 365} \right) = \text{£} \left(\frac{1316}{250} - \frac{366}{250} \right) \times r = \text{£} \frac{19}{5} \times r$$

which agrees with the expression already found above.

EXERCISES 6

Where necessary, answers should be worked out to the nearest penny.

Take 1 year equivalent to 365 days.

Calculate the simple interest on :

1. £304 3s. 4d. for 57 days at $4\frac{1}{2}$ per cent. per annum.
 2. £7434 for $8\frac{3}{4}$ years at $2\frac{1}{2}$ per cent. per annum.
 3. £356 12s. 6d. for 16 days at $2\frac{1}{4}$ per cent. per annum. (R.S.A.)
 4. £62 for 79 days at $3\frac{1}{4}$ per cent. per annum.
 5. £183 12s. 6d. for 7 months at 5 per cent. per annum. (R.S.A.)
 6. £2103 for 2 years 57 days at $3\frac{1}{2}$ per cent. per annum.
 7. £325 for 52 days at $5\frac{1}{2}$ per cent. per annum. (R.S.A.)
 8. £1783 for 304 days at $3\frac{3}{4}$ per cent. per annum.
 9. £8645 for 172 days at $5\frac{1}{4}$ per cent. per annum.
 10. £28,462 for 1 year 60 days at $5\frac{1}{2}$ per cent. per annum.
-
11. (i) Calculate the simple interest at $5\frac{1}{4}$ per cent. per annum on £842 for the period from January 16th, 1935, to May 29th, 1937, inclusive.
- (ii) Interest on a sum of money for 222 days at $3\frac{1}{4}$ per cent. per annum amounts to £6 18s. 9d. What is the sum of money? (C.I.S.)
12. Find the simple interest on £657 for 16 months at $5\frac{1}{2}$ per cent. per annum.
13. What sum of money will amount to £299 15s. 9d. in 9 months at $4\frac{1}{2}$ per cent. per annum?
14. Find the rate per cent. per annum at which, in seven months, the simple interest on £527 will be £17 13s. 6d. (R.S.A.)
15. The simple interest charged on a loan of £320 for 59 days is £2 6s. 7d. Find the rate of interest per cent. per annum. (R.S.A.)
16. To repay a sum of money borrowed five months earlier a man agreed to pay £529 15s. Find the amount borrowed if the rate of interest charged was $4\frac{1}{2}$ per cent. per annum. (L.Ch.C.)
17. A man borrowed £480 on March 12th last at $4\frac{1}{2}$ per cent. per annum on the understanding that the debt should be cleared with interest before it reached £500. What is the latest date of repayment? (R.S.A.)

18. A man borrows £803 for one year. He repays £511 at the end of 164 days, and the balance together with the total interest on the loan at the end of the year. How much interest at $6\frac{1}{4}$ per cent. per annum does he have to pay?

19. On October 31st a man borrowed from his bank an exact number of pounds at the rate of $5\frac{1}{2}$ per cent. per annum. On the following December 31st he owed the bank £464 4s. 7d. How much did he borrow? (R.S.A.)

20. A man borrowed £220 from his bank on October 7th, and on the following December 31st interest £2 8s. 8d. was charged to him. Find the rate of interest per cent. per annum. (R.S.A.)

21. On March 3rd last a man borrowed £380 at $5\frac{1}{2}$ per cent. per annum simple interest. On May 27th he paid back £200. What further sum should he pay on July 31st next to clear the debt? (R.S.A.)

22. On December 3rd, 1938, a man borrowed £575 at simple interest. Up to and including February 6th the rate of interest was $5\frac{1}{2}$ per cent. per annum, but from and including February 7th the rate of interest was raised to $6\frac{1}{2}$ per cent. per annum. What did he pay on June 20th this year to clear the debt with interest? (R.S.A.)

23. A man borrowed £400 from his bank on August 1st and paid off £100 of it on the following September 25th. The interest charged to him on the following December 31st was six guineas. What rate of interest per cent. per annum was charged?

24. A man borrows £1030 from his bank on July 14th and repays £300 on the following September 25th. On December 31st he is charged £19 as interest; find the rate per cent. per annum at which this interest is calculated.

25. On July 1st a boy had £32 16s. in the Post Office Savings Bank. During the next six months he made the following deposits: on July 12th, £1 10s.; on August 7th, £1 17s.; and on November 3rd, 15s. On September 23rd he withdrew £2 and on December 16th he withdrew £1. How much interest does he get for the six months, July-December? The interest is at the rate of $2\frac{1}{2}$ per cent. per annum for every complete pound deposited for a complete month, and begins on the first day of the month following the deposit. Take each month as one-twelfth of a year. (R.S.A.)

26. Instead of paying £127 10s. cash now, a man agrees to pay a certain sum at once and an equal sum a year hence. How much should each sum be, reckoning interest at 4 per cent. per annum? (R.S.A.)

27. A man borrowed £400 from his bank on August 14th and paid off £100 of it on the following October 6th. The interest charged to him on the following December 31st was £5 15s. 11d. What rate of interest per cent. per annum was charged? (R.S.A.)

28. At the end of December last year *A* withdrew £100 from his account in the Post Office Savings Bank and lent it to *B*. It was agreed that the debt should be repaid with interest in four quarterly instalments, each of £26, which *B* is paying this year at the ends of March, June, September and December. As soon as he receives it *A* promptly deposits each repayment in the Post Office Savings Bank, which pays interest at the rate of $2\frac{1}{2}$ per cent. per annum. How much more interest does *A* receive on his loan than he would have done if his £100 had remained all the year in the bank? (R.S.A.)

29. A man borrows £200 from his bank on August 1st and a further sum of £150 on September 10th. The rate of interest charged is at first $5\frac{1}{2}$ per cent. per annum and then changes to $4\frac{1}{2}$ per cent. per annum on October 1st. How much interest does he have to pay on the following December 31st? (R.S.A.)

30. On July 8th a loan of £1679 was repaid with simple interest at the rate of $3\frac{1}{4}$ per cent. per annum by a cheque for £1697 13s. 9d. On what day was the money borrowed?

31. The principal £*P* which amounts to a given sum £*A* in *t* years at *r* per cent. per annum, simple interest, is given by the formula :

$$P = \frac{100 \times A}{100 + r \times t}.$$

Use this formula to calculate the principal which amounts to £50 in two years at 4 per cent. per annum. (U.L.C.I.)

32. A man invested part of £729 for two years at $3\frac{3}{4}$ per cent. per annum and the remainder for three years at $4\frac{1}{4}$ per cent. per annum. The simple interest gained was the same in each case. Find the separate amounts invested.

33. In 1939, *X* borrows £350 on March 5th and a further £250 on May 13th. He repays £400 on July 3rd. Reckoning simple interest at $3\frac{1}{4}$ per cent. per annum, calculate the sum he must pay on September 14th to settle the debt and interest completely.

CHAPTER VII

APPLICATIONS OF SIMPLE INTEREST. BILLS OF EXCHANGE, DISCOUNT, AVERAGE DATE

7.1. Kinds of Discount.

A DEFINITION of discount in connection with profit and loss transactions has already been given in Section 5.6 (page 78). In the settlement of large accounts, especially in international trade, there are other forms of discount which will now be considered.

Generally, there are three kinds of discount according to the nature of the transaction involved. These are :

- (i) **Trade Discount**, already defined and dealt with in Chapter V.
- (ii) **Banker's or Commercial Discount**, used in commerce mainly in connection with Bills of Exchange.
- (iii) **True Discount**, which is seldom used in practice and which is therefore chiefly of theoretical interest.

As (i) has previously been dealt with, it remains to consider (ii) and (iii).

7.2. Bills of Exchange.

In the Bills of Exchange Act, 1882, a Bill of Exchange is defined as "*An unconditional order in writing, addressed by one person to another, signed by the person giving it, requiring the person to whom it is addressed to pay, on demand, or at a fixed or determinable future time, a sum certain in money, to or to the order of a specified person or to bearer*".

As an example of the use of a Bill of Exchange, suppose that on March 21st, 1937, *A* buys goods from *B* on the understanding

that he will pay £2015 for them six months later, i.e. on September 21st, 1937. A Bill of Exchange will be drawn up by *A* in which it will be stated that he will pay *B* the agreed sum on September 21st. This Bill will then be given to *B* in lieu of immediate payment.

7·3. Discounting a Bill.

It may happen that *B* requires the money before September 21st, 1937, and, as a consequence, he takes the Bill to a bill-broker and requests him to **discount** it. If the Bill is legally valid and the broker is satisfied that *A* will be able to pay on the specified date, he will give *B* what is known as the **Present** or **Cash Value** of the Bill and then draw the full amount from *A* on September 21st.

It will be evident that the Present or Cash Value (P.V.) on any day between the date the Bill was drawn up and the specified date of payment will be actually less than the amount shewn on the Bill. The difference represents the broker's charge for carrying out the transaction and is calculated as the simple interest on the sum specified in the Bill for the unexpired time at the current bank rate. This interest is known as **Banker's** or **Commercial Discount**.

7·4. Days of Grace.

In olden times the communication between towns was not so rapid as it is to-day and, as a consequence, three days were added to the date upon which a Bill was due in order to allow time to send the payment. This law is still in force and, in commercial transactions, these three "**Days of Grace**", as they are called, must always be added. Hence, the Bill used as an illustration in Section 7·2 legally becomes due on September 24th, i.e. three days after September 21st.

To exemplify the method of calculation, the above Bill will now be set out in arithmetical form.

Ex. 1. *A Bill for £2015 is drawn on March 21st, 1937, at six months and is discounted on the following July 6th at $2\frac{1}{4}$ per cent. per annum. Calculate (i) the Banker's Discount and (ii) the Cash Value of the Bill on July 6th.*

(i) The Bill is nominally due six months after March 21st, i.e. on September 21st, but it is not legally due until three days later, i.e. on September 24th.

Since it is discounted on July 6th, the number of days between this date and September 24th

$$= 25 + 31 + 24 = 80.$$

Hence, the Banker's Discount

$$= \text{S.I. on } £2015 \text{ for } 80 \text{ days at } 2\frac{1}{4}\% \text{ per annum}$$

$$= £ \frac{2015 \times 80 \times 2\frac{1}{4}}{100 \times 365} = £ \frac{2015 \times 80 \times 9}{100 \times 365 \times 4} = £ \frac{403 \times 9}{5 \times 73}$$

$$= £ \frac{3627}{365} = £9 \text{ } 18\text{s. } 9\text{d. to the nearest penny.}$$

(ii) Knowing now the Banker's Discount, the Cash Value of the Bill on July 6th

$$= £2015 - £9 \text{ } 18\text{s. } 9\text{d.}$$

$$= £2005 \text{ } 1\text{s. } 3\text{d.}$$

Note that the person who held the Bill actually gets £9 18s. 9d. less than the full amount specified because he needed payment before September 24th, whilst the broker draws £2015 on that date and therefore benefits to the extent of £9 18s. 9d. This represents his fee for undertaking the transaction.

Ex. 2. *A Bill at four months, drawn on February 1st, was discounted on April 10th at 2 per cent. per annum for £8369 14s. Find the amount of the Bill.*

Bill nominally due on June 1st.

„ legally due on June 4th.

From April 10th to June 4th there are

$$(20 + 31 + 4) \text{ days} = 55 \text{ days.}$$

Now let the amount of the Bill be $\pounds B$, then the Cash Value on April 10th = $\pounds B - (\text{S.I. on } \pounds B \text{ for 55 days at } 2\% \text{ per annum})$

$$= \pounds \left(B - \frac{B \times 55 \times 2}{100 \times 365} \right) = \pounds \left(1 - \frac{11}{3650} \right) \times B = \pounds \frac{3639}{3650} \times B.$$

$$\therefore \frac{3639}{3650} \times B = 8369.7.$$

$$\text{Hence, } B = \frac{8369.7 \times 3650}{3639} = 23 \times 365 = 8395,$$

so that the Bill was for $\pounds 8395$.

This problem might have been solved by finding the Cash Value of $\pounds 100$ on April 10th and then, by proportion, determining what sum of money would have the given Cash Value on this date.

EXERCISES 7A

Calculate the Banker's Discount on a Bill for :

1. $\pounds 608$ 6s. 8d. drawn on January 15th at 11 months and discounted on May 13th at $6\frac{3}{4}$ per cent. per annum.

2. $\pounds 1277$ 10s. drawn on July 4th at three months and discounted on July 19th at $3\frac{3}{4}$ per cent. per annum.

3. $\pounds 425$ drawn on March 21st at seven months and discounted on August 12th at $2\frac{1}{2}$ per cent. per annum.

4. $\pounds 209$ 17s. 6d. drawn on June 8th at six months and discounted on September 2nd at $3\frac{1}{2}$ per cent. per annum.

5. $\pounds 816$ drawn on March 18th at eight months and discounted on July 19th at $2\frac{1}{4}$ per cent. per annum.

6. $\pounds 1252$ drawn on January 8th at ten months and discounted on May 25th at $2\frac{1}{2}$ per cent. per annum.

7. $\pounds 12,000$ drawn on January 15th at eleven months and discounted on September 26th at $3\frac{1}{4}$ per cent. per annum.

8. 2701 dollars drawn at eight months on March 7th and discounted on August 15th at 4 per cent. per annum. Give the discount in English money, taking 4.64 dollars to the \pounds .

9. 140,525 francs drawn on October 23rd, 1937, at six months and discounted on February 17th, 1938, at $2\frac{1}{4}$ per cent. per annum. Give the answer in English money, taking $178\frac{1}{2}$ francs to be equivalent to $\pounds 1$.

10. £10,835 drawn on April 15th at nine months and discounted at Oslo on October 10th at $3\frac{3}{4}$ per cent. per annum. Give the discount in kroner, the exchange at the time being 19·71 kroner to the £.

11. A Bill for £438 drawn on January 15th at 11 months was discounted on May 13th by the cash payment of £420 15s. Find the rate per cent. per annum charged.

12. A Bill for £580 was drawn on April 5th at six months' date and discounted on the following June 7th at the rate of $2\frac{1}{4}$ per cent. per annum. Find for what sum the Bill was discounted. (R.S.A.)

13. A Bill for £520, drawn on April 8th at 90 days' date, was discounted for £515 16s. 8d. at the rate of $4\frac{1}{2}$ per cent. per annum. On what day was it discounted? (R.S.A.)

14. A Bill for £1700, drawn on March 21st at seven months, was discounted at $2\frac{1}{2}$ per cent. per annum for £1691 10s. Find the date upon which it was discounted.

15. A Bill for £420 was drawn on August 10th at 60 days' date and discounted on September 17th for £418 8s. 4d. Find the rate of interest per cent. per annum charged. (R.S.A.)

16. Find the present worth of a Bill for £415, drawn June 18th at 60 days and discounted on July 10th at $6\frac{1}{2}$ per cent. per annum. (R.S.A.)

17. The Cash Value of a Bill for £912 10s., drawn on February 15th at eight months, is £904 5s., the discount rate being $2\frac{3}{4}$ per cent. per annum. Find the date upon which the Bill was discounted.

18. Find the banker's discount on a Bill for £430, drawn on March 17th at 60 days and discounted on April 13th at $4\frac{1}{2}$ per cent. per annum. (R.S.A.)

19. The Cash Value of a Bill for £657 drawn on March 15th at 8 months and discounted on September 24th is £653 5s. 9d. Find the discount rate of interest per cent. per annum.

20. A Bill, drawn on April 13th at 6 months, was discounted at a bank on May 23rd at $4\frac{1}{4}$ per cent. per annum. The discount was £18 4s. 2d. What was the value of the Bill? (U.L.C.I.)

21. A Bill for 16,936 dollars, drawn on April 8th at three months, was discounted at $3\frac{1}{8}$ per cent. per annum in New York on June

18th, no days of grace being allowed. The proceeds were sent to London, the rate of exchange being 4.64 dollars to the £. Find the discounted value in English money.

22. Find, at $5\frac{1}{2}$ per cent. per annum, the discountable value, on September 12th, of a Bill for £240 drawn on August 3rd at 60 days' date. (R.S.A.)

23. Three Bills for £574, £325 and £380 respectively were discounted at the same time by a banker at 4 per cent. per annum. The Bills were legally due in 65, 38 and 23 days respectively. What was the total discount? (U.L.C.I.)

24. The holder of a bill of exchange for £1000 gets it discounted at 6 per cent. per annum three months before it is legally due. If he were to invest the proceeds immediately at 6 per cent. per annum, simple interest, by how much would the amount in three months' time fall short of £1000? (R.S.A.)

25. *A* holds a Bill from *B* for £925 drawn on April 20th and payable 60 days after date. *B* holds a Bill from *A* for £1200 drawn on May 12th and payable 90 days after date. The accounts are settled on June 1st by *A* giving *B* a Bill payable 70 days after date. Reckoning discount at 4 per cent. per annum, find the amount of the Bill. (L.Ch.C.)

26. *A* owes 2117 dollars to *B* in New York, payable on July 10th, and he holds an accepted Bill for 97,900 francs payable in Paris on August 12th. If both Bills are discounted on June 1st at $3\frac{3}{4}$ per cent. per annum, no days of grace being allowed, find *A*'s credit balance on that date in English money, taking £1 = 4.64 dollars = 178 francs.

7.5. True Discount.

It has already been stated that what is called *True Discount* is seldom used in practice, but it may be interesting to discover exactly what it is and why it is so called. Perhaps the best way will be to discuss a suitable example.

Ex. 3. *X gives Y a Bill of Exchange for £1130 drawn on March 28th at nine months. Y discounts the Bill on August 7th at $3\frac{3}{4}$ per cent. per annum and invests the proceeds immediately at $3\frac{3}{4}$ per cent. per annum simple interest. How much less will Y have on the date the Bill was legally due than he would have had if it had not been discounted?*

Here the Cash Value on August 7th must first be found.

The Bill is nominally due on December 28th and legally due on December 31st.

Hence, the unexpired time is the number of days from August 7th to December 31st. This is

$$(24 + 30 + 31 + 30 + 31) \text{ days} = 146 \text{ days.}$$

\therefore Banker's Discount on August 7th

= S.I. on £1130 for 146 days at $3\frac{3}{4}\%$ per annum,

$$= \pounds \frac{1130 \times 146 \times 15}{100 \times 365 \times 4} = \pounds \frac{339}{20} = \pounds 16 \text{ 19s.}$$

\therefore Cash Value on August 7th = £1130 - £16 19s. = £1113 1s.

Now, since Y invests this sum for 146 days at $3\frac{3}{4}\%$ per annum, the S.I. gained

$$= \pounds \frac{22261 \times 146 \times 15}{20 \times 100 \times 365 \times 4} = \pounds \frac{66783}{4000} = \pounds 16 \text{ 13s. 11d.,}$$

to the nearest penny.

This interest is less than the banker's discount on August 7th by £16 19s. - £16 13s. 11d. = 5s. 1d.,

so that, by discounting the Bill, Y loses 5s. 1d.

It is evident that whilst Y loses, the bill-broker gains, but if the broker's discount had been calculated so that it gave the same sum as the simple interest gained by investing the corresponding Cash Value for the unexpired period at the same rate, such discount would be *True Discount*.

7-6. Correspondence of Terms.

The amount specified in a Bill is known as its **Face Value** and the true cash value is usually referred to as the **Present Value**, whilst the simple interest on the Present Value is called the **True Discount**. Comparing these terms with those used in simple interest, it will be seen that, when a Bill is discounted on a date before it is legally due at a given rate per cent. per annum,

- (i) the **Present Value** corresponds to the **Principal**,
- (ii) the **True Discount** corresponds to the **Simple Interest**, and
- (iii) the **Face Value** corresponds to the **Amount**.

Ex. 4. *A Bill for £673, drawn on May 17th at seven months, was discounted on October 8th at $4\frac{3}{4}$ per cent. per annum on the principle of True Discount. Find the discount allowed.*

The Bill is nominally due on December 17th and legally due on December 20th. Hence, from October 8th to December 20th, there are $(23 + 30 + 20)$ days = 73 days.

Now remembering that the Face Value of the Bill represents the Amount of the Present Value for 73 days at $4\frac{3}{4}\%$ per annum, it is advisable to start with £100 as the Present Value.

S.I. on £100 for 73 days at $4\frac{3}{4}\%$ per annum

$$= \pounds \frac{100 \times 73 \times 19}{100 \times 365 \times 4} = \pounds \frac{19}{20} = \pounds 0.95.$$

\therefore Amount of £100 for 73 days at $4\frac{3}{4}\%$ per annum = £100.95.

Hence, on October 8th, True Discount on £100.95 = £0.95.

$$\begin{aligned} \therefore \text{True Discount on } \pounds 673 &= \pounds \frac{0.95 \times 673}{100.95} = \pounds 6\frac{1}{3} \\ &= \pounds 6 \text{ 6s. 8d.} \end{aligned}$$

7.7. Difference between True and Banker's Discount.

Although mainly of theoretical interest, it will assist in gaining a clear understanding of the whole subject of practical discount and interest if the nature of the difference between these two forms of discount is known.

At a given rate per cent. per annum and on a day before a Bill is legally due, the

$$\begin{aligned}
 \text{Banker's Discount} &= \text{S.I. on the Bill,} \\
 &= \text{S.I. on (Present Value + True Discount)} \\
 &= \text{S.I. on Present Value} + \text{S.I. on True Dis-} \\
 &\quad \text{count} \\
 &= \text{True Discount} + \text{S.I. on True Discount,}
 \end{aligned}$$

$$\therefore \text{Banker's Discount} - \text{True Discount} = \text{S.I. on True Discount,}$$

i.e. the difference between Banker's and True Discount is equal to the simple interest on the True Discount, or, the Banker's Discount is equal to the Amount at simple interest of the True Discount.

Hence, in practice, the bill-broker's commission for discounting a Bill is greater than the True Discount by the simple interest on the True Discount.

Ex. 5. *A Bill is discounted at $6\frac{1}{4}$ per cent. per annum four months before it is legally due and the difference between the Banker's and True Discount is 7s. 1d. Calculate (i) the amount of the Bill, (ii) the Cash Value and (iii) the Present Value.*

(i) Since the difference between the Banker's and True Discount is equal to the simple interest on the True Discount,

$$\therefore \text{S.I. on True Discount} = 7\text{s. } 1\text{d.} = \pounds \frac{17}{48}.$$

Hence, if £ D be the True Discount,

$$\frac{D \times 4 \times 25}{100 \times 12 \times 4} = \frac{17}{48}$$

so that

$$D = \frac{17 \times 100 \times 12 \times 4}{48 \times 4 \times 25} = 17.$$

\therefore True Discount = £17,

and Banker's Discount = £17 + 7s. 1d. = £17 7s. 1d.

But this is the interest on the Bill for 4 months at $6\frac{1}{4}\%$ per ann.,

$$\therefore \frac{B \times 4 \times 25}{100 \times 12 \times 4} = \frac{833}{48},$$

where £ B is the amount of the Bill.

Hence
$$B = \frac{833 \times 100 \times 12 \times 4}{48 \times 4 \times 25} = 833,$$

\therefore the amount of the Bill was £833.

(ii) The Cash Value = Bill - Banker's Discount

$$= £833 - £17 \text{ 7s. 1d.} = £815 \text{ 12s. 11d.}$$

(iii) The Present Value = Bill - True Discount

$$= £833 - £17 = £816.$$

EXERCISES 7B

Take a year to be 12 months or 365 days according as the time is given in months or days.

Find the True Discount in each of the following cases :

No.	Amount of bill	Legally due in	Rate % per annum
1.	£275 12s	16 months	$4\frac{1}{2}$
2.	£1017	32 days	$3\frac{1}{2}$
3.	£370 15s. 6d.	210 days	$2\frac{3}{4}$
4.	£463	85 days	$2\frac{3}{4}$
5.	£2718	250 days	$4\frac{3}{4}$

6. A Bill is discounted at $2\frac{1}{2}\%$ per cent. per annum 100 days before it becomes legally due, and the True Discount is £24 6s. 8d. Calculate (i) the Banker's Discount and (ii) the amount of the Bill.

7. The True Discount on a Bill of £7351 on a date 60 days before it is legally due is £51. Calculate the rate per cent. per annum at which the discount has been calculated.

8. On August 5th a loan of £520 was repaid with interest at the rate of $3\frac{1}{2}$ per cent. per annum, by a cheque for £523 12s. 10d. On what day was the money borrowed? (R.S.A.)

9. The Present Value of a Bill for £917 15s., legally due in 56 days, is £912 10s. Find the rate per cent. per annum at which the True Discount has been calculated.

10. The difference between the Banker's and True Discounts on a Bill discounted at 4 per cent. per annum 125 days before it became legally due is 2s. 3d. Find (i) the amount of the Bill, (ii) the Cash Value and (iii) the Present Value.

11. Find, to the nearest penny, the difference between the Banker's Discount and the True Discount on a Bill of £650 13s., legally due in three months, the rate being $4\frac{1}{4}$ per cent. per annum. (U.L.C.I.)

12. *B* holds a Bill from *A* which is legally due in 80 days. He agrees to take immediate payment at $4\frac{1}{4}$ per cent. per annum True Discount, but *A* refuses. *B* accordingly gets the Bill discounted by a banker at $6\frac{1}{4}$ per cent. per annum and thus loses £12 10s. 5d. Find (i) the amount of the Bill, (ii) the True Discount and (iii) the Banker's Discount.

13. *P* holds a Bill for £2555 which he discounts on March 1st at $2\frac{3}{4}$ per cent. per annum, thus receiving £2542 9s. 9d. Find the date upon which the Bill was legally due, and calculate the True Discount on March 1st.

14. Calculate the rate per cent. per annum at which the True Discount on a Bill legally due in ten months' time will be exactly the same as the Banker's Discount at $7\frac{1}{2}$ per cent. per annum.

7·8. Settling Several Accounts.

In business, it frequently happens that several accounts are due from a trader *A* to another trader *B*. Instead of paying these separately, it is generally more convenient for *A* to settle them all at the same time. The date upon which this is to be done must then be determined.

Consider a concrete case.

Ex. 6. *A* received the following accounts for three months from a manufacturer *B*, each subject to three months' credit.

January 29th.	To goods supplied	-	-	£74.
February 17th.	„ „ „	-	-	£142.
March 2nd.	„ „ „	-	-	£316.

Calculate the date of settlement most advantageous to *A*.

Since *B* allows three months' credit, the accounts will be actually due on April 29th, May 17th, June 2nd respectively.

Now, it is evident that payment will not be made before April 29th, but suppose it is made d days after that date; interest should then be charged on the overdue account, and if the rate per cent. per annum be denoted by r , then the interest on £74 for d days at $r\%$ per annum is

$$£ \frac{74 \times d \times r}{100 \times 365}.$$

For the second item, the number of days from April 29th to May 17th is $1 + 17 = 18$; hence this account will be only $(d - 18)$ days overdue, and the interest on £142 for $(d - 18)$ days at $r\%$ per annum is

$$£ \frac{142 \times (d - 18) \times r}{100 \times 365}.$$

Finally, for the third item, the number of days from April 29th to June 2nd is $1 + 31 + 2 = 34$, so that this account will be $(d - 34)$

days overdue ; the interest is therefore that on £316 for $(d - 34)$ days at $r\%$ per annum

$$\pounds \frac{316 \times (d - 34) \times r}{100 \times 365}.$$

Now the most advantageous date to A will be given when d is chosen so that the total interest due has the least value ; but the least value will be zero, hence the value of d will be given by the equation :

$$\frac{74 \times d \times r}{100 \times 365} + \frac{142 \times (d - 18) \times r}{100 \times 365} + \frac{316 \times (d - 34) \times r}{100 \times 365} = 0.$$

Each term has the common factor $\frac{r}{100 \times 365}$, by which the equation may be divided throughout ; then

$$74 \times d + 142 \times (d - 18) + 316 \times (d - 34) = 0,$$

or $74 \times d + 142 \times d - 2556 + 316 \times d - 10744 = 0 ;$

$$\therefore (74 + 142 + 316) \times d = 2556 + 10744 = 13300,$$

i.e. $532 \times d = 13300.$

Hence, $d = \frac{13300}{532} \dots\dots\dots(a)$

$$= 25,$$

so that the most convenient date will be 25 days after April 29th,

i.e. **May 24th.**

Note that the first amount is due on April 29th, and if paid on May 24th, i.e. 25 days later, interest should be charged on the overdue account. Taking $\pounds k$ as the interest on $\pounds 1$ for one day at an agreed rate per cent. per annum, this interest will be

$$\pounds (74 \times 25 \times k) = \pounds (1850 \times k).$$

Similarly, the settlement of the second amount will be 7 days late, and the interest due on this will be $\pounds (142 \times 7 \times k) = \pounds (994 \times k) ;$

\therefore total interest due on the two overdue accounts will be

$$\pounds (1850 + 994) \times k = \pounds (2844 \times k).$$

Now the third amount is not due until June 2nd, and if paid on May 24th discount should be allowed for the unexpired period between the two dates, i.e. for 9 days. This discount is

$$£(316 \times 9 \times k) = £(2844 \times k).$$

This is just equal to the interest chargeable on the two overdue accounts, so that a settlement on May 24th is a just one for both *A* and *B*. The determination of the appropriate date for just settlement is therefore very important, and such date is usually known as the **Average or Mean Date Due**.

7.9. Practical Method of finding the Average Date Due.

A close study of the solution to Ex. 6 will soon shew that it is unnecessary to carry out the calculation at such length. The equation, when reduced to its final form (*a*), consists in dividing the number 13300 by the sum of the amounts due. But the number 13300 is really the sum of the products of each amount and the corresponding number of days from the date the first payment becomes due. Hence, a simple practical method of calculation may readily be deduced.

The date upon which the first payment falls due is known as the **zero date**, so that, in Ex. 6, the zero date is April 29th. The second payment of £142 is due on May 17th, which is 18 days later, whilst the third payment of £316 falls due on June 2nd, or 34 days after the zero date. With this information, the working may then be set out simply as follows.

Taking the products of amounts and the corresponding periods in days :

$$\begin{array}{rcl} 74 \times 0 & = & 0 \\ 142 \times 18 & = & 2556 \\ 316 \times 34 & = & 10744 \\ \hline \therefore \text{Sum of products} & = & 13300 \end{array}$$

and sum of amounts due = £(74 + 142 + 316) = £532.

Hence, if the average date due is d days after the zero date,

$$532 \times d = 13300,$$

or

$$d = \frac{13300}{532} = 25.$$

By this simple process the same final stage has been reached with the minimum of working. The practical rule may therefore be enunciated as follows :

First find the zero date and the periods, in days, from this date on which the several accounts fall due ; then divide the sum of the products of each amount and its corresponding period in days by the total of the amounts due. The quotient gives the number of days from the zero date to the Average Date Due.

Ex. 7. Calculate the date on which the total of the following amounts should be paid in order to make a complete settlement :

March 28.	To Goods	-	-	£461,	Credit—1 month.
April 15.	,,	-	-	£447,	,, 1 month.
April 27.	,,	-	-	£529,	,, 2 months.
May 5.	,,	-	-	£1417,	,, 3 months.

The date of complete settlement is clearly the Average Date Due. Now the several payments fall due on April 28th, May 15th, June 27th, August 5th respectively ; hence the zero date is April 28th.

Number of days from April 28th to

- (i) May 15th = $2 + 15 = 17$,
- (ii) June 27th = $2 + 31 + 27 = 60$,
- (iii) August 5th = $2 + 31 + 30 + 31 + 5 = 99$.

Hence the following products of amounts and their corresponding periods :

$$\begin{aligned} 461 \times 0 &= 0 \\ 447 \times 17 &= 7599 \\ 529 \times 60 &= 31740 \\ 1417 \times 99 &= 140283 \end{aligned}$$

$$\therefore \text{Sum of products} = \overline{179622}$$

and the sum of amounts due = £2854.

∴ Number of days from zero date to that of settlement

$$= \frac{179622}{2854} = 62.93 \dots = 63,$$

so that the date of settlement will be 63 days after April 28th,
i.e. on **June 30th.**

7·10. The Settlement of a Balance.

The principle of determining the average date due of a single account of several items may also be extended to the settlement of a balance in an account between two traders. An example should be sufficient to shew how the application may be carried out practically.

Ex. 8. *Two traders, X, Y, buy goods from one another, each allowing the other one month's credit. At the end of three months the accounts rendered are as follows :*

Goods sold by X to Y				Goods sold by Y to X					
<i>April</i>	18	-	-	£63	<i>April</i>	23	-	-	£52
<i>May</i>	15	-	-	£71	<i>May</i>	24	-	-	£49
<i>June</i>	16	-	-	£79					

Calculate the date upon which the balance should be paid so that no interest is due either to X or to Y.

Since credit for one month is allowed, Y's payments to X fall due on May 18, June 15, July 16 respectively, whilst X's payments to Y become due on May 23 and June 24.

Taking May 18 as the zero date, the number of days from this date to

- (i) June 15 = 13 + 15 = 28,
- (ii) July 16 = 13 + 30 + 16 = 59,
- (iii) May 23 = 5,
- (iv) June 24 = 13 + 24 = 37.

Hence the products of amounts and their corresponding periods are

For Y's payments	For X's payments
$63 \times 0 = 0$	$52 \times 5 = 260$
$71 \times 28 = 1988$	$49 \times 37 = 1813$
$79 \times 59 = 4661$	$\quad = 2073$
\therefore Sums of products = <u>6649</u>	
<u>2073</u>	

Excess of Y's products

over X's - - = 4576

But total of Y's liabilities to X = £(63 + 71 + 79) = £213

and „ X's „ „ Y = £(52 + 49) = £101

\therefore Balance due to X - - - - = £112

\therefore Number of days from the zero date to that of settlement is

$$\frac{4576}{112} = 40.857 \dots = 41.$$

Hence the date of settlement of the balance is 41 days after May 18, i.e. on **June 28th.**

This means that on June 28th Y has to pay £112 to X to clear the account.

EXERCISES 7c

1. The following quarterly account was received by a trader, each item being subject to three months' credit. Calculate the average date of settlement.

July	26.	To goods supplied	-	-	-	£112
August	10.	„ „ „	-	-	-	£83
September	8.	„ „ „	-	-	-	£225

2. Find the date upon which a complete settlement must be made for the total sum due in the following account :

July	26.	To goods	-	-	£53, Credit—2 months.
August	12.	„ „	-	-	£47, „ 2 „
August	18.	„ „	-	-	£61, „ 2 „
September	19.	„ „	-	-	£103, „ 2 „

3. Calculate the date of complete settlement for the total of the following amounts :

June 28th.	Goods	-	-	£617,	Credit—2 months.
July 5th.	„	-	-	£533,	„ 2 „
July 14th.	„	-	-	£1381,	„ 2 „
August 2nd.	„	-	-	£1234,	„ 3 „

4. Calculate the date on which the total of the following amounts should be paid so as to make a complete settlement :

March 5.	Goods	-	-	£153,	Credit—2 months.
March 29.	„	-	-	£276,	„ 2 „
April 16.	„	-	-	£152,	„ 2 „
May 25.	„	-	-	£209,	„ 1 month.

(R.S.A.)

5. Find the average date due for the following account :

February 24	-	-	-	£183,	Credit—2 months.
March 7	-	-	-	£237,	„ 2 „
April 6	-	-	-	£342,	„ 2 „
April 19	-	-	-	£879,	„ 3 „

6. The mean due date of payment of four bills was 10th June. Three of the bills were payable as follows :

£418 on 29th April, £323 on 20th May, £551 on 3rd June.

The fourth bill was for £1007 ; on what date was it due ? (U.L.C.I.)

7. A firm *P* owes a manufacturer *Q* the following amounts, due on the specified dates :

April 11th	-	-	-	-	£632 10s.
May 9th	-	-	-	-	£416 18s.
June 11th	-	-	-	-	£327 16s.

Find the date upon which *P* should make a complete settlement.

8. Two traders *A* and *B* buy goods from one another and, at the end of the first three months of 1937, their accounts were :

Sold to <i>B</i> from <i>A</i>				Sold from <i>B</i> to <i>A</i>			
January 8.	Goods	-	£53	January 26.	Goods	-	£60
February 20.	„	-	£76	February 19.	„	-	£83
March 10.	„	-	£61				

Calculate the date upon which the settlement of the balance should be made so that no interest may be chargeable on either side, credit being allowed for one month on each item.

9. Calculate the date of complete settlement of the following account between two firms :

<i>Dr.</i>				<i>Cr.</i>			
April 15.	To Goods	-	£127	May 4.	By Goods	-	£132
May 3.	„ „	-	£159	May 16.	„ „	-	£147
May 14.	„ „	-	£216	June 3.	„ „	-	£119
May 26.	„ „	-	£283				

Credit allowed—2 months.

10. A merchant has two bills to pay to the same firm, one for £1387, dated March 14th, and the other for £1679, dated April 26th, the credit in each case being three months. It is agreed that settlement should be made on the average date due, but the merchant pays the total account on June 12th and is allowed banker's discount at $2\frac{3}{4}$ per cent. per annum. Find (i) the average date due for settlement, and (ii) the discount allowed.

11. Calculate the average date for the settlement of the balance of the following account between two traders *H* and *K* :

Sold by <i>H</i> to <i>K</i>			Sold by <i>K</i> to <i>H</i>		
March 25.	Goods	- £116	April 11.	Goods	- £123
April 1.	„	- £132	April 20.	„	- £62
April 9.	„	- £58	May 2.	„	- £113
April 21.	„	- £117	May 11.	„	- £137
May 6.	„	- £231			
May 17.	„	- £73			

Credit on each item—2 months.

12. Determine the average date for the settlement of the balance of the following account between two manufacturers *P* and *Q* :

Sold by <i>P</i> to <i>Q</i>			Sold by <i>Q</i> to <i>P</i>		
January 28.	Goods	- £140	February 9.	Goods	- £151
February 6.	„	- £84	February 21.	„	- £97
February 15.	„	- £113	March 4.	„	- £89
February 27.	„	- £67	March 10.	„	- £119
March 8.	„	- £129			
March 26.	„	- £157			

Credit allowed on each item—3 months.

13. A dealer bought goods to the following amounts on the stated dates, one month being allowed in each case for payment : £187 on March 15th, £94 on April 10th, £108 on May 1st and £205 on May 27th. He paid £250 on May 14th. On what date would the balance of £344 be an equitable settlement of his account ?
(L.Ch.C.)

14. A London firm bought goods from a New York manufacturer to the following amounts, each with a credit for two months from the specified date of purchase :

March 18th, £483 ; April 19th, £523 5s. ; May 9th, £362 5s.
Calculate (i) the average date of settlement, and (ii) the equivalent of the whole debt in dollars when the exchange rate is 4.68 dollars to the £.

CHAPTER VIII

COMPOUND INTEREST AND DEPRECIATION

8.1. Compound Interest.

IN calculating interest for more than a year, it is the practice to add the interest to the principal at the end of each year to give a new principal for the following year. Thus, suppose £100 to be invested for three years at $2\frac{1}{2}$ per cent. per annum ; then

Interest at end of first year = $2\frac{1}{2}\%$ of £100 = £2 10s. ;

∴ £100 becomes £102 10s. at the end of one year, and this is the principal for the beginning of the second year ; hence

Interest at end of second year = $2\frac{1}{2}\%$ of £102 10s. = £2 11s. 3d.,

so that the principal at the beginning of the third year is £105 1s. 3d. Again, interest at end of 3rd year = $2\frac{1}{2}\%$ of £105 1s. 3d. = £2 12s. 6d., to the nearest penny. Therefore, in three years £100 amounts to £107 13s. 9d., so that the actual interest, by this principle, on £100 for 3 years at $2\frac{1}{2}$ per cent. per annum is £7 13s. 9d.

Obviously, the process may then be continued for any number of years.

Interest calculated in this way is known as **Compound Interest**, and is briefly denoted by the letters **C.I.**

In direct calculations it is convenient to work to *five* places of decimals of £1, to ensure accuracy to the nearest penny. The method is applied in the following illustrative example.

Ex. 1. Calculate, to the nearest penny, the compound interest on £527 for three years at 4 per cent. per annum.

Whilst it is advisable to work to five places of decimals, in

general, especially when the given principal involves shillings and pence, four places will suffice when the interest has to be found on a whole number of pounds, as in the present case.

Since $4\% = 0\cdot04$, each year's interest may be found by multiplying the principal for that year by 4 and setting down each digit of the product two places to the right of the digit multiplied and ignoring all figures that would occupy places greater than four. The working then appears as follows :

$$\begin{array}{rcl}
 £527\cdot0000 & = & \text{Principal.} \\
 \underline{21\cdot0800} & = & \text{Interest for the first year.} \\
 548\cdot0800 & = & \text{Amount at end of first year.} \\
 \underline{21\cdot9232} & = & \text{Interest for the second year.} \\
 570\cdot0032 & = & \text{Amount for two years.} \\
 \underline{22\cdot8000} & = & \text{Interest for the third year.} \\
 592\cdot8032 & = & \text{Amount for three years.} \\
 \underline{527\cdot0000} & = & \text{Original Principal.} \\
 65\cdot8032 & = & \text{C.I. for three years} \\
 & = & £65\ 16s. \ 1d., \text{ to the nearest penny.}
 \end{array}$$

Had all the digits in the several products been retained, the C.I. would have been £65·803328, which is larger than that already found by $£0\cdot000128 = 0\cdot03072$ pence, so that the rejected digits do not affect the practical answer.

8·2. Calculation of Interest Half-yearly.

Sometimes the interest is added half-yearly ; then the calculation is carried out by adding half the year's interest at the end of each six months, as is shewn in Ex. 2.

Ex. 2. *A man invests £546 17s. 6d. at $4\frac{1}{2}$ per cent. compound interest. What will it amount to in two years if the interest is added half-yearly ?*

If each £100 is to earn $£4\frac{1}{2}$ in one year, it will earn $£2\frac{1}{4}$ in six months ; hence the interest may be calculated in the same way

as if the rate per cent. per annum had been $2\frac{1}{4}$ and the time 4 years.

In finding the interest for each half-year, it is convenient to use two lines, one giving the interest at 2% and the other giving $\frac{1}{4}$ %. The 2% is obtained by multiplying the principal by two and writing each digit in the product two places to the right, whilst the $\frac{1}{4}$ % is found either by dividing the 2% line by 8 or the principal by 400. The working then appears as follows :

Since 17s. 6d. = $\pounds\frac{7}{8}$ = £0.875, the principal is £546.8750.

£546.87500 = Principal.

$\begin{array}{r} 10.93750 \\ 1.36719 \end{array} \}$ = Interest for the first half-year.

559.17969 = Amount for the first half-year.

$\begin{array}{r} 11.18358 \\ 1.39795 \end{array} \}$ = Interest for the second half-year.

571.76122 = Amount at end of one year.

$\begin{array}{r} 11.43522 \\ 1.42938 \end{array} \}$ = Interest for the third half-year.

584.62582 = Amount for the third half-year.

$\begin{array}{r} 11.69250 \\ 1.46156 \end{array} \}$ = Interest for the fourth half-year.

597.77988 = Amount for two years

= £597 15s. 7d., to the nearest penny.

Sometimes interest is payable quarterly and, in that case, a similar method of calculation to the above is used. Thus, to find the C.I. on a sum of money, payable quarterly, for n years at r per cent. per annum, the interest per cent. *per quarter* is $\pounds\frac{1}{4}r$ and the number of quarters is $4n$; hence, the required interest must be calculated at $\frac{1}{4}r$ per cent. per quarter for $4n$ quarters.

8.3. Compound Interest Tables.

When a large number of compound interest calculations have to be made, specially prepared books of tables are generally used. The following gives a very brief extract from such tables.

The amount of £1 at Compound Interest.

Years	2½%	3%	3½%	4%	4½%
1	1·02500000	1·03000000	1·03500000	1·04000000	1·04500000
2	1·05062500	1·06090000	1·07122500	1·08060000	1·09202500
3	1·07689063	1·09272700	1·10871787	1·12486400	1·14116613
4	1·10381289	1·12550881	1·14752300	1·16985856	1·19251860
5	1·13140821	1·15927407	1·18768631	1·21665290	1·24618194

Ex. 3. Find, by use of the table, the compound interest on £1657 for four years at $3\frac{1}{2}$ per cent. per annum.

Rejecting all decimals of £1 beyond the fifth place, the working may effectively be set out as follows :

For 4 years at $3\frac{1}{2}$ % per annum, the amount of

$$\begin{array}{rcl}
 \text{£} & \text{£} & \text{£} \\
 1000 = 1\cdot147523 \times 1000 = 1147\cdot52300 \\
 600 = \text{,,} \times 600 = 688\cdot51380 \\
 50 = \text{,,} \times 50 = 57\cdot37615 \\
 7 = \text{,,} \times 7 = 8\cdot03266 \\
 \hline
 1657 & & = 1901\cdot44561 \\
 & & \underline{1657\cdot00000} = \text{Principal} \\
 & & 244\cdot44561 = \text{C.I.} \\
 & & = \text{£}244 \text{ 8s. 11d.}
 \end{array}$$

8·4. The General Formula.

The principle of compound interest is so important, as will be seen later in Chap. XVI, that it is necessary to establish a general formula for practical usage.

Suppose a sum of £ P amounts to £ A in n years at r per cent. per annum compound interest, then the relation between P , A , n and r will give the needed formula.

Let i be the interest on £1 for one year at r % per annum, then

$$i = \frac{r}{100}.$$

Very frequently, $1+i$ is denoted by R , so that, for one year at $r\%$ per annum,

$$\text{amount of } \pounds 1 = \pounds(1+i) = \pounds\left(1 + \frac{r}{100}\right) = \pounds R;$$

$$\therefore \text{amount of } \pounds P = \pounds P(1+i) = \pounds PR.$$

Now $\pounds P(1+i)$ becomes the principal for the second year, so that the amount of $\pounds P(1+i)$ for one year at $r\%$ per annum

$$= \pounds P(1+i) \times (1+i) = \pounds P(1+i)^2,$$

i.e. the amount of $\pounds P$ for 2 years at $r\%$ per annum $= \pounds P(1+i)^2$.

Similarly the amount of $\pounds P$ for 3 years at $r\%$ $= \pounds P(1+i)^3$, and so on.

Hence, if n denotes a positive integer, the amount of $\pounds P$ for n years at $r\%$ per annum $= \pounds P(1+i)^n$;

$$\text{i.e.} \quad A = P(1+i)^n = P\left(1 + \frac{r}{100}\right)^n = PR^n.$$

This is the formula required.

Ex. 4. *£781 5s. is invested at 4 per cent. per annum compound interest. What will it amount to in three years?*

$$\text{Here} \quad i = \frac{4}{100} = 0.04,$$

$$\text{so that} \quad 1+i = 1.04.$$

$$\begin{aligned} \therefore \text{Amount in three years} &= \pounds 781.25 \times (1.04)^3 \\ &= \pounds 781.25 \times 1.124864 \\ &= \pounds 878.8 = \pounds 878 \text{ 16s.} \end{aligned}$$

Note that the C.I. gained $= \pounds 878 \text{ 16s.} - \pounds 781 \text{ 5s.} = \pounds 97 \text{ 11s.}$

8.5. Use of the Formula.

A direct application of the formula has been made in Ex. 4, but often the formula is much more useful in solving problems involving compound interest, of which the following are examples.

Ex. 5. *A sum of money amounts to £34,460 10s. in three years and to £35,322 0s. 3d. in four years at compound interest. Find (i) the rate per cent. per annum, and (ii) the sum of money.*

(i) If the sum of money be £ P , then

$$PR^4 = 35322\frac{1}{80} \quad \text{and} \quad PR^3 = 34460\frac{1}{2}.$$

Hence, by division, in order to eliminate the unknown P ,

$$R = \frac{35322\frac{1}{80}}{34460\frac{1}{2}} = \frac{2825761 \times 2}{80 \times 68921} = \frac{41}{40},$$

i.e.

$$1 + \frac{r}{100} = \frac{41}{40}.$$

Taking 1 from each side, $\frac{r}{100} = \frac{1}{40}$, or $r = \frac{100}{40} = 2\frac{1}{2}$;

\therefore the rate = $2\frac{1}{2}\%$ per annum.

(ii) To find P , substitute the value of R just found in one of the above equations, preferably the second, as it involves a lower power of R :

$$P \times \left(\frac{41}{40}\right)^3 = 34460\frac{1}{2},$$

so that

$$P = \frac{68921 \times 40 \times 40 \times 40}{2 \times 41 \times 41 \times 41} = 32000.$$

\therefore the sum of money = £32,000.

Ex. 6. *A man borrows £3783 for three years at 5 per cent. per annum compound interest, on the condition that the complete debt must be repaid in three equal instalments, one at the end of each year. Calculate what each instalment must be.*

In a problem of this type, it is much more convenient to construct a general statement first in symbols and then substitute the particular values given. Thus, denoting the sum borrowed by £ P , the amount at the end of the first year = £ $P(1+i)$ = £ PR .

Let each of the equal instalments be £ S , then the principal left at the beginning of the second year = £ $(PR - S)$, and this amounts to £ $(PR - S) \times (1+i)$ or £ $(PR - S) \times R$, or £ $(PR^2 - SR)$ at the end of the second year.

Another repayment of £ S is now made, so that the principal for the third year = £($PR^2 - SR - S$). At the end of the third year this amounts to £($PR^2 - SR - S$) $\times R$ or £($PR^3 - SR^2 - SR$), and the third repayment of £ S completely liquidates the debt.

$$\therefore PR^3 - SR^2 - SR - S = 0,$$

or
$$(R^2 + R + 1) \times S = P \times R^3.$$

In the problem under consideration,

$$R = 1 + \frac{5}{100} = 1.05, \quad \text{and} \quad P = 3783.$$

Hence, by substitution in the relation just found

$$\{(1.05)^2 + 1.05 + 1\} \times S = 3783 \times (1.05)^3,$$

i.e. $(1.1025 + 1.05 + 1) \times S = 3783 \times (1.05)^3,$

or $3.1525 \times S = 3783 \times 1.157625 ;$

$$\therefore S = \frac{3783 \times 1.157625}{3.1525}$$

$$= \frac{3783 \times 1157625}{3152500} = 1389.15,$$

so that each instalment = £1389.15 = £1389 3s.

8-6. Depreciation.

When a man buys a car and then uses it regularly, without accident, the value of the car at the end of each year will be much less than that at the beginning of the year owing to the wear and tear caused by use. This reduction in value is known as **Depreciation**, and is usually calculated as a percentage on the compound interest principle.

The same is true of buildings, machinery, furniture, and indeed everything that is used. If this were not so, there would practically be no business, and therefore commercial arithmetic would be unnecessary.

The following example will shew the application of the compound interest principle to problems on depreciation.

Ex. 7. *A small factory with equipment costs £7832 and, on the average, £850 is spent annually on maintaining it in good repair. In estimating its value each year, 20 per cent. of its value at the beginning of the year is deducted for depreciation and then the maintenance allowance added. Calculate the value of the factory at the end of four years.*

In making the necessary calculation, the procedure is precisely the same as in Ex. 1 and Ex. 2, with the exception that the depreciation each year must be subtracted instead of being added as in C.I. problems.

The working then appears as follows :

$$\begin{array}{rcl}
 & £ & \\
 7832\cdot0000 & = & \text{Original value of factory and equipment.} \\
 1566\cdot4000 & = & \text{Depreciation at end of first year.} \\
 \hline
 6265\cdot6000 & & \\
 850\cdot0000 & = & \text{Maintenance allowance.} \\
 \hline
 7115\cdot6000 & = & \text{Value at beginning of second year.} \\
 1423\cdot1200 & = & \text{Depreciation at end of second year.} \\
 \hline
 5692\cdot4800 & & \\
 850\cdot0000 & = & \text{Maintenance allowance.} \\
 \hline
 6542\cdot4800 & = & \text{Value at beginning of third year.} \\
 1308\cdot4960 & = & \text{Depreciation at end of third year.} \\
 \hline
 5233\cdot9840 & & \\
 850\cdot0000 & = & \text{Maintenance allowance.} \\
 \hline
 6083\cdot9840 & = & \text{Value at beginning of fourth year.} \\
 1216\cdot7968 & = & \text{Depreciation at end of fourth year.} \\
 \hline
 4867\cdot1872 & & \\
 850\cdot0000 & = & \text{Maintenance allowance.} \\
 \hline
 5717\cdot1872 & = & \text{Value after four years} \\
 & = & \text{£5717 3s. 9d., to the nearest penny.}
 \end{array}$$

8·7. The General Formula for Depreciation.

A formula analogous to that for compound interest may readily be found in the case of depreciation, although generally it is not so frequently used.

If $\pounds D$ be the depreciated value of $\pounds 1$ for one year at r per cent. per annum, then

$$D = 1 - \frac{r}{100},$$

so that $\pounds P$ will in one year depreciate in value to $\pounds PD$.

At the end of the second year this will become $\pounds PD \times D = \pounds PD^2$, and by repeating the process, if $\pounds V_n$ is the value at the end of n years, then

$$V_n = PD^n = P \left(1 - \frac{r}{100} \right)^n.$$

In such an example as that given in Ex. 7, the depreciated value each year is increased by a maintenance allowance. Let $\pounds M$ be this allowance, then in one year,

$$V_1 = PD + M,$$

therefore depreciated value at end of second year

$$= \pounds (PD + M) \times D = \pounds (PD^2 + MD);$$

hence, $V_2 = PD^2 + MD + M$.

Similarly,

$$V_3 = PD^3 + MD^2 + MD + M = PD^3 + (D^2 + D + 1) \times M,$$

$$V_4 = PD^4 + (D^3 + D^2 + D + 1) \times M,$$

and so on.

When n is a positive integer, the value at the end of n years will be given by

$$V_n = PD^n + (D^{n-1} + D^{n-2} + \dots + D + 1) \times M.$$

This formula may be simplified. See Sect. 16·4, p. 242.

To shew the use of this formula, the result already obtained in Ex. 7 may be checked.

$$P = 7832, \quad D = 1 - \frac{1}{5} = \frac{4}{5} = 0.8, \quad M = 850.$$

$$\begin{aligned} \therefore V_4 &= 7832 \times (0.8)^4 + \{ (0.8)^3 + (0.8)^2 + 0.8 + 1 \} \times 850 \\ &= 7832 \times (0.8)^4 + (0.512 + 0.64 + 0.8 + 1) \times 850 \\ &= 7832 \times 0.4096 + 2.952 \times 850 \\ &= 3207.9872 + 2509.2 = 5717.1872. \end{aligned}$$

Hence, the required value at the end of four years

$$= £5717.1872 = £5717 \text{ 3s. 9d.}, \text{ as already found.}$$

EXERCISES 8

Unless stated otherwise, answers in money should be calculated to the nearest penny. Interest should be reckoned half-yearly when the time involves half-a-year.

Calculate the value of x in each of the following cases, interest to be added annually.

No.	Principal	Amount	C.I.	Time	Rate % per annum
1.	£675	—	x	3 years	4
2.	£632 15s.	x	—	3 "	4
3.	£2463 12s.	—	x	3 "	4
4.	£1862	x	—	5 "	$3\frac{1}{2}$
5.	£3235	—	x	4 "	$3\frac{1}{2}$
6.	£48	—	x	3 "	$2\frac{1}{4}$
7.	£1600	x	—	4 "	$2\frac{3}{4}$
8.	£753 10s.	—	x	7 "	3
9.	£575 14s.	x	—	4 "	5
10.	£832 17s. 6d.	x	—	4 "	$4\frac{1}{2}$

11. Find the amount of £945 2s. 6d. in three years at $4\frac{1}{2}$ per cent. per annum compound interest. (U.L.C.I.)

12. Find the increase at compound interest of £319 7s. 6d. for $3\frac{1}{2}$ years at $3\frac{1}{2}$ per cent. per annum. (U.L.C.I.)

13. Find which of (a) and (b) gives the greater amount at compound interest :

(a) £207 in four years at 3 per cent. per annum.

(b) £207 in three years at 4 per cent. per annum.

Find also the difference between the two amounts. (U.L.C.I.)

14. *A* lends *B* £450 on the condition that *B* is to pay interest at 4 per cent. per annum on whatever sum remains to be repaid. *B* pays back £148 at the end of the first year and £156 at the end of the second year. How much must he repay at the end of the third year to clear the whole debt?

15. A sum of £490 13s. 4d. is invested at $2\frac{1}{2}$ per cent. per annum compound interest. What sum will it amount to in two years if the interest is added half-yearly?

16. Calculate the compound interest on £436 for two years at $4\frac{1}{2}$ per cent. per annum, the interest being payable half-yearly.

17. Find, to the nearest shilling, the difference between the simple and compound interest on £1475 for four years at 6 per cent. per annum. (L.Ch.C.)

18. Find the difference between the compound interest and the simple interest on £832 for six years at $4\frac{1}{4}$ per cent. per annum.

19. A sum of £400 is set aside at the beginning of each of three years and bears compound interest at $4\frac{1}{2}$ per cent. per annum. What will be the total amount at the end of the third year?

20. (i) If compound interest at 5 per cent. per annum is reckoned half-yearly, express as a percentage per annum the actual rate of compound interest reckoned yearly.

(ii) Calculate the compound interest on £83 10s. for two years at 5 per cent. per annum, reckoned half-yearly. (R.S.A.)

21. A man deposited £300 in a savings bank at the end of 1934. The money was left to accumulate under compound interest at 3 per cent. per annum, payable half-yearly. What was the amount at the end of 1936? (R.S.A.)

22. On February 24th, 1932, a man opened an account in the Post Office Savings Bank with a deposit of £75. Since that time he has made no further deposit and no withdrawal. Every February 24th he sends his pass book to the Controller, who returns it with the interest added up to the preceding December 31st. What credit balance does the pass book shew at the present time (March 23rd, 1937)? The Bank pays interest at the rate of $2\frac{1}{2}$ per cent. per annum on whole numbers of £s only for whole months only: i.e. no interest is paid on any part of a £ or for any part of a month. (R.S.A.)

23. A sum of money amounts to £2704 in two years at 4 per cent. per annum; what would it have amounted to had the rate per cent. per annum been $2\frac{1}{2}$?

24. The compound interest on £3 for two years is five shillings; what will be the compound interest on £12 for four years at the same rate of interest?

25. A certain sum of money was invested at $4\frac{1}{2}$ per cent. per annum compound interest. It was found that the increase during the third year exceeded the increase during the second year by exactly £6. Find the sum invested. (U.L.C.I.)

26. If the simple interest on £500 for three years is £67 10s., find the compound interest on the same sum for the same time at the same rate of interest per annum. (L.Ch.C.)

27. In two years a sum of money will amount to £507 at 4 per cent. per annum compound interest. To what would it amount at 5 per cent. in two years? (U.L.C.I.)

28. A sum of £6643 is borrowed for three years at $3\frac{3}{4}$ per cent. per annum compound interest on the condition that it must be paid back in three equal instalments, one at the end of each year. Calculate the value of each instalment.

29. A man borrowed £820 3s. 4d. for three years at $2\frac{1}{2}$ per cent. per annum compound interest on the condition that he repays the complete debt in three equal instalments, one at the end of each year. Find the amount of each instalment.

30. £3281 is divided into two shares so that, when invested at $2\frac{1}{2}$ per cent. per annum compound interest, the amount of one share at the end of two years has to be equal to the amount of the other share at the end of four years. Find the shares.

31. A sum of £230 is invested on the first day of 1927, and an equal sum on the first days of 1928 and 1929. Find the amount at the end of 1929, allowing compound interest at $4\frac{1}{2}$ per cent. per annum, convertible yearly. Calculate also the single sum which, invested on January 1st, 1927, would at the same rate reach the same amount as the answer at the end of 1929. (C.I.S.)

32. A motor car is purchased for £830 and each year its value depreciates by 23 per cent. of its value at the beginning of the year. What will the car be worth at the end of four years?

33. A manufacturer bought some machinery for £6750, and it was estimated that every year the machinery depreciated by 7 per cent. of its value at the beginning of the year. At the end of four years the manufacturer sold the plant for £5025. Did he gain or lose on the estimated value, and by how much?

34. A manufacturer bought machinery for £6650. It was estimated that every year this machinery depreciated by 6 per cent. of its value at the beginning of that year. If he sold the machinery at the end of five years for £4500, how much of the value estimated at that time did he lose? (R.S.A.)

35. To allow for the depreciation of machinery costing £750, 15 per cent. of its cost is deducted at the end of the first year; at the end of the second and of each succeeding year, $7\frac{1}{2}$ per cent. of its value at the beginning of the year is deducted. What is its value, to the nearest £, after four years' use? (L.Ch.C.)

36. To allow for depreciation of a house costing £1000, 15 per cent. of its value is deducted at the end of the first year; $12\frac{1}{2}$ per cent. of its value at the beginning of the second year is deducted at the end of that year, and at the end of each succeeding year 8 per cent. is deducted. After how many years will its value be just less than half its original cost and what will then be its value?

37. The initial cost of a factory and its equipment is £10,000 and £800 is spent annually on repairs and maintenance. In estimating their value at the end of each year, 20 per cent. is deducted for depreciation and the maintenance cost then added. What will be the estimated value at the end of five years?

38. To allow for the depreciation in the value of machinery costing £800, at the end of each year 12 per cent. of its value at the beginning of the year is deducted in the accounts. After how many years will its value first be entered as less than £400, and what will then be its value to the nearest shilling? (L.Ch.C.)

39. A machine purchased in 1927 had its original cost entered in books of the firm buying it. Each year onwards, the book value was determined by deducting a fixed percentage from the value at the beginning of the year. In 1931, the book value was £648, and in 1932, it was £486. Calculate (i) the fixed percentage deducted, and (ii) the original cost.

40. The depreciation in value of a small manufacturing plant is calculated each year by deducting a fixed percentage of the value at the beginning of the year. The value at the end of the second year is recorded as £2083 4s., and the deduction made for that year is £520 16s. Calculate (i) the percentage deducted, (ii) the original value of the plant, and (iii) its estimated value at the end of five years.

CHAPTER IX

STOCKS AND SHARES

9.1. Raising Capital.

IN business, it is essential to have sufficient money with which to trade. This money is known as **Capital**, although all capital is not necessarily actual money, for the value of buildings, equipment and stock must be considered in estimating capital. In large undertakings, no one person, or even a few, could subscribe enough money to float the business, hence a **Joint Stock Company** may be formed under the provisions of the Companies Act of 1929, which governs especially the issue of capital. Without going into more detail than is necessary for a clear understanding of the relevant arithmetical operations, it should suffice to state that the capital required by a company is divided into **shares** which members of the community are invited to buy as a good investment. Those who purchase shares are called **shareholders**, and the total number of shares, or their value, held by any shareholder is referred to as his **holding**. The profit made by a company during a given period, usually one year, is frequently devoted to (i) building up a Reserve Fund, and (ii) paying interest to the shareholders for the use of their money. This interest is called **dividend** and is declared as a percentage of the capital.

9.2. Classes of Shares.

There are several kinds of shares, but for arithmetical purposes only two need be considered. These are :

- (i) **Preference Shares** on which a fixed rate of dividend is usually paid out of profits before any becomes payable to the ordinary shareholders.

- (ii) **Ordinary Shares** upon which such dividend is payable after the prior claims on the profits have been met. Thus, an ordinary shareholder's dividend may vary in rate from year to year.

When a permanent or redeemable loan has been made to a company needing further capital, owing perhaps to an extension of business, the acknowledgment of such a debt is known as a **Debenture**. This carries a regular and generally a fixed rate of interest.

9.3. The Buying and Selling of Shares.

Just as the prices of most goods vary according to market conditions, so the prices of shares are also subject to fluctuations. Thus, a £10 share may be sold for as much as £15 when, for instance, the dividend prospect is good, whilst, on the other hand, the price may fall to £8, or even lower, when business is slack. There are, however, many conditions which affect the prices of investments, but it will not be necessary for these to be considered here.

Fully paid-up shares are often represented by **stock** and the prices of shares are generally quoted for a nominal value of £100, which is called **£100 stock**. For example, a stock quoted as a *3½ per cent. at 104* really means that the price of a share nominally worth £100 is £104, and that this carries with it a dividend of £3 10s. per annum.

It should be observed that whilst only an integral number of shares can be purchased, where the capital is issued in shares, yet any nominal value of stock may be bought. In solving problems, however, it is often convenient to regard the nominal value of a stock as a *share* of £100. In this sense it is possible to buy fractions of a share.

When the market value of a share is equal to its nominal value, it is said to be **at par**, but when the price is above the nominal value, it is said to be **above par** or **at a premium**, and when the price is below the nominal value it is said to be **below par** or **at a discount**.

The actual percentage return on the total money invested is known as the **Yield**. This must not be confused with the dividend, i.e. the interest on each share of paid-up stock. Thus, for an investment of £ S in a $c\%$ stock at S , the dividend = £ c , but the yield = $100\ c/S$ per cent. Consider carefully the following examples.

Ex. 1. *A man buys 680 shares at five guineas each in a company X and 700 shares at £6 16s. each in a company Y. How much money does he invest altogether? If, for a certain year, X declares a dividend of $4\frac{1}{4}\%$ per cent. and Y a dividend of $2\frac{1}{2}\%$ per cent., calculate*

(i) *his total income for that year, and*

(ii) *the total percentage return on his investment.*

Sum invested in X = 680 shares at £ $5\frac{1}{4}$ = £3570.

„ „ Y = 700 „ „ £ $6\frac{4}{5}$ = £4760.

∴ Total sum invested = £8330.

$$(i) \text{ Income from X} = \pounds \frac{3570 \times 4\frac{1}{4}}{100} = \pounds \frac{3570 \times 17}{100 \times 4} = \pounds \frac{6069}{40}$$

$$= \pounds 151\frac{29}{40}$$

$$= \pounds 151\ 14s.\ 6d.$$

$$\text{Income from Y} = \pounds \frac{4760 \times 2\frac{1}{2}}{100} = \pounds \frac{4760 \times 5}{100 \times 2}$$

$$= \pounds 119.$$

$$\therefore \text{Total income} = \pounds 151\ 14s.\ 6d. + \pounds 119$$

$$= \pounds 270\ 14s.\ 6d.$$

$$(ii) \pounds 270\ 14s.\ 6d. = \pounds 270\frac{29}{40}.$$

∴ Percentage return on the total investment

$$= \frac{270\frac{29}{40} \times 100}{8330} = \frac{10829 \times 100}{40 \times 8330} = \frac{13}{4} = 3\frac{1}{4}\%.$$

Ex. 2. *The capital of a company is made up, in nominal shares of £100, of 3256 Ordinary shares, 2352 Preference shares and 152 debenture-holders. The profits available for interest and dividends in a certain year were £35,418. The interest due to the debenture-holders was $4\frac{1}{2}$ per cent. and the dividend payable on the Preference shares was $7\frac{1}{2}$ per cent. Calculate the percentage dividend due to the ordinary shareholders.*

Since the nominal value of each share and debenture is £100,

$$\therefore \text{Interest due to debenture-holders} = £152 \times 4\frac{1}{2} = £684,$$

and dividend due to holders of Preference shares

$$= £2352 \times 7\frac{1}{2} = £17,640.$$

Hence, the total amount due to holders of debentures and Preference shares = £684 + £17,640 = £18,324.

\therefore Amount available for Ordinary shareholders

$$= £35,418 - £18,324 = £17,094.$$

Hence, if the percentage dividend payable to these is r ,

$$3256 \times r = 17094,$$

so that

$$r = 17094/3256 = 21/4 = 5\frac{1}{4},$$

i.e. the dividend for Ordinary shareholders is at the rate of $5\frac{1}{4}\%$.

Ex. 3. *A man invests £1161 in a $4\frac{1}{2}$ per cent. stock at $96\frac{3}{4}$ and £1449 in a $3\frac{3}{4}$ per cent. stock at $103\frac{1}{2}$. Find (i) the number of shares he bought in each stock, and (ii) his total annual income from the investments, after paying income tax at 5s. 6d. in the £.*

In the first stock, a share nominally worth £100 costs $£96\frac{3}{4}$;

$$\therefore \text{Number of shares purchased} = \frac{1161}{96\frac{3}{4}} = \frac{4 \times 1161}{387} = 12.$$

In the second stock, the price of £100 share is $103\frac{1}{2}$;

$$\therefore \text{Number of shares bought} = \frac{1449}{103\frac{1}{2}} = \frac{2 \times 1449}{207} = 14.$$

Since each share in the $4\frac{1}{2}\%$ stock brings a dividend of $£4\frac{1}{2}$ and each share in the $3\frac{3}{4}\%$ stock brings a dividend of $£3\frac{3}{4}$,

$$\therefore \text{Total gross income} = £(4\frac{1}{2} \times 12) + £(3\frac{3}{4} \times 14) = £54 + £52\frac{1}{2} = £106\frac{1}{2}.$$

Now a tax of 5s. 6d. in the £ leaves 14s. 6d. or $\text{£}\frac{29}{40}$ as the equivalent of each £ of gross income ;

$$\therefore \text{Net income} = \text{£}106\frac{1}{2} \times \frac{29}{40} = \text{£}\frac{213 \times 29}{2 \times 40} = \text{£}\frac{6177}{80} = \text{£}77\frac{17}{80}$$

$$= \text{£}77 \text{ 4s. 3d.}$$

Now number of shares held in the $4\frac{3}{4}\%$ stock

$$= \frac{3 \times P}{361} = \frac{3 \times 15884}{361} = 132.$$

On selling these at $99\frac{3}{4}$, he gets £($99\frac{3}{4} \times 132$), and the number of shares purchased in the $5\frac{1}{2}\%$ at $104\frac{1}{2}$ is

$$\frac{99\frac{3}{4} \times 132}{104\frac{1}{2}} = \frac{399 \times 132 \times 2}{4 \times 209} = 126.$$

But in this stock the number of shares already held is

$$\frac{P}{418} = \frac{15884}{418} = 38.$$

∴ Total number of shares now held = $126 + 38 = 164$, so that his income = £($5\frac{1}{2} \times 164$) = £902 ;

∴ Increase in income = £902 - £836 = £66.

9.4. The Method of Comparison.

A prospective investor often wishes to know which of two stock quotations represents the better investment. To make an effective comparison, the yield in each case may be found or, alternatively, the same sum may be supposed to be invested in each stock and the resulting incomes compared. Both methods are illustrated in Ex. 5.

Ex. 5. *The quotations of two stocks are (i) 4 per cent. at $104\frac{3}{4}$; (ii) $3\frac{3}{4}$ per cent. at $97\frac{1}{2}$. Determine which represents the better investment.*

First Method. Comparison of the Yields.

Working to two places of decimals, the percentage yield of

$$(i) \text{ is } \frac{4 \times 100}{104\frac{3}{4}} = \frac{4 \times 100 \times 4}{419} = \frac{1600}{419} = 3.82.$$

$$(ii) \text{ is } \frac{3\frac{3}{4} \times 100}{97\frac{1}{2}} = \frac{15 \times 100 \times 2}{4 \times 195} = \frac{50}{13} = 3.85.$$

Hence, the $3\frac{3}{4}\%$ at $97\frac{1}{2}$ represents the better investment.

Second Method. Investing the same amount in each stock.

To render the calculation as short as possible, take the product of the two prices as the amount ; then £($104\frac{3}{4} \times 97\frac{1}{2}$) invested in

(i) the 4% earns, as dividend, £($97\frac{1}{2} \times 4$) = £390,

(ii) the $3\frac{3}{4}$ % earns, as dividend, £($104\frac{3}{4} \times 3\frac{3}{4}$) = £392,

Hence, the $3\frac{3}{4}$ % at $97\frac{1}{2}$ again proves to be the better investment.

9.5. The Method of Mixtures.

No wise person keeps all his eggs in one basket, and the same principle applies to investments. When a man has a certain sum to invest, he frequently puts part of it in one stock and the remainder in another, provided he can secure an adequate return for his money. Consider this problem generally.

Suppose a man invests a sum of £ S partly in a per cent. stock at A and partly in b per cent. stock at B so that his total dividend is £ i . Then, if £ P be invested in the a % and £ Q in the b %, $P+Q=S$ and the total dividend £ i is equal to

$$£\frac{P \times a}{A} + £\frac{Q \times b}{B}.$$

Now let c be the total percentage yield from the two investments, then $\frac{100 \times i}{S} = c$, or $\frac{100 \times i}{P+Q} = c$, so that $i = \frac{(P+Q) \times c}{100}$.

$$\text{Hence,} \quad i = \frac{P \times a}{A} + \frac{Q \times b}{B} = \frac{(P+Q) \times c}{100}.$$

Multiply out by 100 :

$$\frac{P \times a \times 100}{A} + \frac{Q \times b \times 100}{B} = (P+Q) \times c.$$

But $\frac{a \times 100}{A}$ is the percentage yield of the a % stock and $\frac{b \times 100}{B}$ is the percentage yield of the b % stock ; denoting these yields by p and q respectively, and assuming $p > c > q$, the above equation becomes

$$\begin{aligned}
 & P \times p + Q \times q = (P + Q) \times c, \\
 \text{or} \quad & P \times p - P \times c = Q \times c - Q \times q; \\
 \text{i.e.} \quad & P \times (p - c) = Q \times (c - q); \\
 & \therefore \frac{P}{Q} = \frac{c - q}{p - c} \text{ or } \frac{q - c}{c - p}.
 \end{aligned}$$

$$\text{Hence, } \frac{\text{Sum invested in } a\%}{\text{Sum invested in } b\%}$$

$$= \frac{\text{Difference between total and } b\% \text{ yields}}{\text{Difference between total and } a\% \text{ yields}}.$$

$$\frac{S}{P} = \frac{P + Q}{P} = 1 + \frac{p - c}{c - q} = \frac{p - q}{c - q},$$

$$\therefore P = \frac{c - q}{p - q} \text{ of } S, \text{ and similarly, } Q = \frac{p - c}{p - q} \text{ of } S.$$

If $p < c < q$; then

$$P = \frac{q - c}{q - p} \text{ of } S \quad \text{and} \quad Q = \frac{c - p}{q - p} \text{ of } S.$$

This rule admits of very simple application, which may conveniently be set out in the following way, assuming $p > c > q$:

Stocks	$a\%$ at A	$b\%$ at B
Percentage Yield	p	q
Total Yield	c	
Ratio	$(c - q) : (p - c).$	

$$\therefore \text{Sum invested in } a\% = \frac{c - q}{p - q} \text{ of } \text{£}S,$$

$$\text{and sum invested in } b\% = \frac{p - c}{p - q} \text{ of } \text{£}S.$$

$$\text{Note that } \frac{c - q}{p - q} + \frac{p - c}{p - q} = \frac{c - q + p - c}{p - q} = 1.$$

Ex. 6. £8874 is to be invested partly in the $3\frac{3}{4}\%$ per cent. at $101\frac{1}{4}$ and partly in the $5\frac{1}{4}\%$ per cent. at $76\frac{1}{2}$ so that a total annual dividend of £493 may be obtained. Find how much must be invested in each stock.

From the general case just discussed, it is evident that the percentage yields must first be found. Now the yield from

$$(i) \text{ the } 3\frac{3}{4}\% \text{ stock} = \frac{3\frac{3}{4} \times 100}{101\frac{1}{4}} = \frac{15 \times 100}{405} = \frac{100}{27} = 3\frac{1}{2}\frac{9}{9},$$

$$(ii) \text{ the } 5\frac{1}{4}\% \text{ stock} = \frac{5\frac{1}{4} \times 100}{76\frac{1}{2}} = \frac{21 \times 100}{306} = \frac{350}{51} = 6\frac{44}{51},$$

$$(iii) \text{ the whole investment} = \frac{493 \times 100}{8874} = \frac{50}{9} = 5\frac{5}{9}.$$

Hence,

$$\begin{array}{cc} 3\frac{3}{4}\% \text{ at } 101\frac{1}{4} & 5\frac{1}{4}\% \text{ at } 76\frac{1}{2} \\ \hline 3\frac{1}{2}\frac{9}{9} & 6\frac{44}{51} \end{array}$$

$$5\frac{5}{9}$$

Ratio

$$1\frac{4}{13} \frac{7}{3} : 1\frac{2}{7} \frac{3}{2} = 12 : 17.$$

\therefore Amount invested in $3\frac{3}{4}\%$ stock = $\frac{12}{29}$ of £8874 = £3672.

and amount invested in $5\frac{1}{4}\%$ stock = $\frac{17}{29}$ of £8874 = £5202.

Large fractions may sometimes be avoided by using income instead of yield. Thus, suppose the whole sum on £8874 to be invested in each stock; then the income obtainable from

$$(i) \text{ the } 3\frac{3}{4}\% \text{ stock} = \text{£}\{3\frac{3}{4} \times 8874 / 101\frac{1}{4}\} = \text{£}328\frac{2}{3},$$

$$(ii) \text{ the } 5\frac{1}{4}\% \text{ stock} = \text{£}\{5\frac{1}{4} \times 8874 / 76\frac{1}{2}\} = \text{£}609.$$

Hence,

$$\begin{array}{cc} 3\frac{3}{4}\% \text{ at } 101\frac{1}{4} & 5\frac{1}{4}\% \text{ at } 76\frac{1}{2} \\ \hline 328\frac{2}{3} & 609 \end{array}$$

$$493$$

Ratio

$$116 : 164\frac{1}{3} = 12 : 17, \text{ as above.}$$

9-6. Brokerage.

The market for stocks and shares is known as a **Stock Exchange**, and the business in connection with the purchase and sale of shares is transacted through an agent called a **Stock-broker**, who is a member of the Stock Exchange. Like other agents, the stock-broker charges a small commission, called **brokerage**, for the service rendered. This commission varies according to the class of stock dealt with. In buying shares, therefore, the cost to the purchaser is slightly increased, and in selling shares the money realised is less in amount by the rate of brokerage chargeable.

In arithmetical problems the rate of commission is generally stated as a percentage of the nominal value of a share when brokerage has to be taken into account, but in many cases the quoted prices of shares frequently include all brokerage and other charges.

Ex. 7. *Find the change in income obtained by selling out £3500 of $4\frac{1}{4}$ per cent. stock at 107 and investing the proceeds in a $2\frac{1}{2}$ per cent. stock at $59\frac{1}{4}$, brokerage at one-eighth per cent. being charged on each transaction.*

Selling out £3500 of $4\frac{1}{4}\%$ stock indicates that the number of shares sold, of nominal value £100, is 35.

Hence, annual income before selling out = $£4\frac{1}{4} \times 35 = £148$ 15s.

Now each share is sold for £107 less the brokerage, i.e. for $£107 - £\frac{1}{8} = £106\frac{7}{8}$; hence cash realised = $£106\frac{7}{8} \times 35$.

This is invested in shares costing $£59\frac{1}{4}$ + brokerage, i.e.

$$£59\frac{1}{4} + £\frac{1}{8} = £59\frac{3}{8}.$$

$$\therefore \text{Number of shares purchased} = (106\frac{7}{8} \times 35) / 59\frac{3}{8},$$

and the annual income obtainable from this investment

$$= £ \frac{106\frac{7}{8} \times 35 \times 2\frac{1}{2}}{59\frac{3}{8}} = £ \frac{855 \times 35 \times 5}{475 \times 2} = £ \frac{315}{2} = £157$$
 10s.

Hence, the income is increased by

$$£157$$
 10s. - $£148$ 15s. = $£8$ 15s.

9-7. Official Stock Quotations.

The actual buying and selling of shares is carried out by a **Stock-jobber**, who is also a member of the Stock Exchange. He quotes the prices for buying and selling, and the difference between these is called the turn of the market and represents a measure of the jobber's profit.

Official quotations are issued twice daily and are published in the newspapers. The following is a sample of such a list :

British Funds			Home Rails		
Consols, $2\frac{1}{2}\%$	-	$71\frac{1}{4}$ - $72\frac{1}{4}$	L.M.S., 4% Deb.	-	$88\frac{1}{2}$ - $90\frac{1}{2}$
War Loan, $3\frac{1}{2}\%$	-	$97\frac{3}{4}$ - $98\frac{1}{4}$	G.W., 4% Deb.	-	97 -99
Funding, 3%	-	$96\frac{1}{4}$ - $97\frac{1}{4}$	S.R., 5% Pref.	-	88 -90

The prices shewn represent the jobbers' quotations at the end of each half-day as dated. Thus, for the day on which the above quotations appeared, the jobber would buy the 3% Funding Loan at $96\frac{1}{4}$ and sell it for $97\frac{1}{4}$. Similarly, his buying price for Southern Railway 5% Preference Shares would be 88, whilst his selling price would be 90. In general, any quotation such as $92\frac{1}{4}$ - $92\frac{3}{4}$ means that a person buying £100 stock would have to pay $\pounds 92\frac{3}{4}$ for it, but if he were selling, he would receive $\pounds 92\frac{1}{4}$ for it, these amounts being exclusive of the broker's charges.

Ex. 8. *From an investment in a $2\frac{1}{2}$ per cent. stock, a man derives an annual income of £450. By selling part of this stock when it was quoted at $80\frac{3}{8}$ - $80\frac{3}{4}$ and investing the proceeds in a $4\frac{1}{2}$ per cent. stock, quoted at $92\frac{3}{4}$ - $93\frac{1}{2}$, he increased his yearly income by £38. Reckoning brokerage at $\frac{1}{8}$ per cent. in each case, find how much of the $2\frac{1}{2}$ per cent. stock he sold.*

Here the selling price of the $2\frac{1}{2}\%$ shares = $\pounds(80\frac{3}{8} - \frac{1}{8}) = \pounds 80\frac{1}{4}$, and the purchase price of the $4\frac{1}{2}\%$ shares = $\pounds(93\frac{1}{2} + \frac{1}{8}) = \pounds 93\frac{5}{8}$.

$$\therefore \text{Ratio of these prices} = \frac{80\frac{1}{4}}{93\frac{5}{8}} = \frac{321 \times 8}{4 \times 749} = \frac{6}{7}.$$

Hence, the cash realised on selling 7 shares of the $2\frac{1}{2}\%$ stock will buy 6 shares of the $4\frac{1}{2}\%$ stock.

Now the original income from the $2\frac{1}{2}\%$ stock = £450.

$$\therefore \text{Number of shares held} = \frac{450}{2\frac{1}{2}} = 180.$$

If he sells 7 of these and buys 6 of the $4\frac{1}{2}\%$ stock,

$$\text{new income} = \text{£}(173 \times 2\frac{1}{2}) + \text{£}(6 \times 4\frac{1}{2}) = \text{£}432\frac{1}{2} + \text{£}27 = \text{£}459\frac{1}{2}.$$

$$\therefore \text{Increase in income} = \text{£}459\frac{1}{2} - \text{£}450 = \text{£}9\frac{1}{2}.$$

But the actual increase is £38 ; so that the number of groups even shares sold = $38 \div 9\frac{1}{2} = 4$,

i.e. the actual number of shares sold = $4 \times 7 = 28$.

\therefore He sold 28 shares or £2800 stock of the $2\frac{1}{2}\%$.

The problem may best be solved by algebra as follows.

Let number of $2\frac{1}{2}\%$ shares sold be x , then proceeds of the sale of these = $\text{£}80\frac{1}{4} \times x$, and the number of shares purchased in the $4\frac{1}{2}\%$

$$= \frac{80\frac{1}{4} \times x}{93} = \frac{6 \times x}{7}.$$

$$\therefore \text{Annual income from these} = \frac{6 \times x \times 4\frac{1}{2}}{7} = \frac{27 \times x}{7}.$$

But in the $2\frac{1}{2}\%$ stock, he now holds only $(180 - x)$ shares so that the income derived from these = $\text{£}(180 - x) \times 2\frac{1}{2}$.

$$\text{Hence, } \frac{27 \times x}{7} + \frac{(180 - x) \times 5}{2} = 450 + 38 = 488.$$

Multiply out by $7 \times 2 = 14$; then

$$54x + 35(180 - x) = 488 \times 14,$$

or

$$54x + 6300 - 35x = 6832,$$

i.e.,

$$54x - 35x = 6832 - 6300$$

$$\therefore 19x = 532$$

so that

$$x = 28.$$

EXERCISES 9

*Where necessary, answers should be given to the nearest penny.
Brokerage is only to be taken into account where stated.*

1. A man possesses 560 shares of 5s. each in a company. A dividend of $8\frac{3}{4}$ per cent. was paid. What did the man receive, income tax at 4s. 9d. in the £ having been deducted? (R.S.A.)

2. A man bought 500 shares at 17s. $7\frac{1}{2}$ d. per share and, later on, 750 more of the same shares at 17s. 9d. per share. After receiving a dividend of 9d. per share he sold out, receiving 17s. $0\frac{1}{2}$ d. for each share. How much money did he gain or lose by the whole transaction? (R.S.A.)

3. A man buys 120 shares at five guineas each. At the end of the year he receives a dividend of 2s. 6d. per share together with one additional share for every ten already held. These new shares he sells at £5 6s. 3d. each. Calculate the percentage return of his total receipts on the original investment.

4. The £15 shares of a company are quoted at £18 15s. and an investor buys 224 shares at this price. Six months later he received a dividend of £56 19s. 3d. after income tax at 4s. 6d. in the £ had been deducted. Calculate the percentage dividend declared by the company.

5. A man buys 84 shares at five guineas each bearing interest at $3\frac{3}{4}$ per cent. per annum, in one company, and 56 shares at £8 15s. each, bearing interest at $2\frac{3}{4}$ per cent. per annum, in another company. Calculate the average percentage return on his total investment.

6. A man bought £300 of $3\frac{1}{2}$ per cent. stock at 105 $\frac{5}{8}$ and also £600 of $4\frac{1}{2}$ per cent. stock at 117 $\frac{3}{8}$. Find, to the nearest penny, the average interest received on each £100 invested. (R.S.A.)

7. From the following investments the dividends stated were received :

Stock	Investment	Annual Dividend
A at $93\frac{3}{4}$	£1781 5s.	£66 10s.
B at $92\frac{1}{4}$	£2121 15s.	£63 5s.
C at $97\frac{1}{2}$	£2827 10s.	£123 5s.

Calculate (i) the percentage dividend declared on each investment, (ii) the total percentage yield for the year.

8. An investor bought 4 per cent. industrial stock at $91\frac{1}{2}$. Find, to the nearest penny, the percentage rate of interest he obtained on his investment (i) without deduction of income tax, (ii) after deducting income tax at 4s. 9d. in the £.

If, instead of buying 4 per cent. industrial stock at $91\frac{1}{2}$, the investor had bought $2\frac{1}{2}$ per cent. railway stock, his percentage rate of interest after deduction of income tax at 4s. 9d. in the £ would have been smaller by 4s. 2d. What was the market price of the railway stock? (U.L.C.I.)

9. A man invested £7715 in a $5\frac{1}{2}$ per cent. stock at 103. After receiving the first year's dividend, the price of the stock rose to 114 and he sold out. He then invested the proceeds in a $4\frac{3}{4}$ per cent. stock at 95; by how much did he increase his income?

10. By how much does the percentage yield of a $4\frac{3}{4}$ per cent. stock at 114 exceed that of a $3\frac{1}{4}$ per cent. stock at $97\frac{1}{2}$?

11. How much money must a man invest in $3\frac{1}{2}$ per cent. stock at $101\frac{1}{4}$ in order to obtain an income of £156 per annum? (R.S.A.)

12. A man invests £6545 in a $3\frac{1}{2}$ per cent. stock at $93\frac{1}{2}$. When the price of the stock falls to $90\frac{3}{4}$ he sells out and invests the proceeds in a $3\frac{3}{4}$ per cent. stock at $96\frac{1}{4}$. Find by how much he increases his annual income.

13. By buying 3 per cent. Government stock at a certain price I find that I obtain 3.6 per cent. for my money and derive a net income from it of £334 16s., after income tax at the rate of 4s. 6d. in the £ has been deducted at the source. Find the amount of stock I hold and the price at which I bought. (C.I.S.)

14. What is the price of a $4\frac{1}{2}$ per cent. stock which gives the same return on capital as a $3\frac{3}{4}$ per cent. stock at 95?

A man sells his holding in the $3\frac{3}{4}$ per cent. stock and invests the proceeds in the $4\frac{1}{2}$ per cent. stock when the price has fallen to $104\frac{1}{2}$. What is his new income if his original income was £110? (L.Ch.C.)

15. Two-thirds of £3420 are invested in a $3\frac{1}{4}$ per cent. stock and the remaining third in a $2\frac{3}{4}$ per cent. stock at $104\frac{1}{2}$. The total income derived is £106; find the price of the $3\frac{1}{4}$ per cent. stock.

16. A man invested £3395 in a $3\frac{1}{2}$ per cent. stock at 97. He sold £2000 of the stock when the price had risen to 104 and invested the proceeds in a 4 per cent. stock at 96. Find his new income. (L.Ch.C.)

17. A man invested three-quarters of his capital in a 4 per cent. stock at 85 and the remainder in a 5 per cent. stock at 102. Find the percentage return on his money, correct to two places of decimals.

His total capital was £4080. If he sold his holding in the 4 per cent. stock when it had risen to $93\frac{1}{2}$ and invested in the 5 per cent. stock, what would be the change in his income, neglecting expenses? (L.Ch.C.)

18. What sums of money must be invested in each of the following investments in order that an income of £100 shall be obtained?

- (i) $3\frac{1}{2}$ per cent. stock at $105\frac{7}{8}$, brokerage $\frac{1}{4}$ per cent.
- (ii) 5 per cent. stock at $118\frac{1}{2}$, brokerage $\frac{1}{2}$ per cent.
- (iii) 5s. shares at 28s. $1\frac{1}{2}$ d., brokerage $1\frac{1}{2}$ d. per share, dividend at 25 per cent.

Hence write down the investment which gives the greatest yield. (R.S.A.)

19. A man invested £4536 in a $3\frac{1}{2}$ per cent. stock at $70\frac{3}{4}$ and, after receiving the first year's dividend, sold out and invested the proceeds in a $5\frac{1}{4}$ per cent. stock at $98\frac{3}{4}$, brokerage in each case being one-eighth per cent. Find the change in his income.

20. Find the change in income produced by selling out £3500 of $5\frac{1}{2}$ per cent. stock at 107 and investing the proceeds in $2\frac{1}{2}$ per cent. Consols at $58\frac{3}{4}$. Brokerage $\frac{1}{8}$ per cent. on each transaction. (R.S.A.)

21. *W* sells £8250 stock of the 3 per cent. at $101\frac{5}{8}$ and invests the proceeds in a $3\frac{3}{4}$ per cent. stock at 103, brokerage at $\frac{1}{8}$ per cent. being charged on each transaction. By how much does *W* increase his annual income after deducting tax at 5s. 6d. in the £?

22. A man wishes to invest £6324 partly in a $3\frac{1}{4}$ per cent. stock at $94\frac{1}{4}$ and partly in a $5\frac{1}{2}$ per cent. stock at $107\frac{1}{4}$ in order to secure an annual income of £279. How much must he invest in each stock?

23. It is desired to secure an annual income of £55 by investing £936 partly in a $6\frac{1}{4}$ per cent. stock at 104 and partly in a $5\frac{1}{4}$ per cent. stock at 91. How much must be invested in each stock?

24. After paying income tax at 5s. 6d. in the £, a man had £51 9s. 6d. left from the income derived from investments in a $4\frac{3}{4}$ per cent. stock at 91 and a $5\frac{1}{2}$ per cent. stock at 112. The total sum invested was £1400; find how much was invested in each stock.

25. A man invests part of his capital in $3\frac{1}{2}$ per cent. debentures at $97\frac{1}{2}$ and the remainder in $5\frac{1}{2}$ per cent. preference stock at $126\frac{1}{2}$ in order to get a return of exactly 4 per cent. on the money invested. Find how much he must invest in each stock if his capital is £6375. (U.L.C.I.)

26. £2987 is invested partly in $3\frac{1}{2}$ per cent. at 87 and partly in $4\frac{1}{2}$ per cent. at 116 so that an annual income of £118 is obtained. How much was invested in each stock?

27. A man invests equal sums in a $3\frac{1}{2}$ per cent. stock at $87\frac{1}{2}$ and in a 6 per cent. stock at $112\frac{1}{2}$. Income tax and expenses reduce his gross income by 5s. in the £. If his net income is £441, find the amount invested in each stock. (L.Ch.C.)

28. A man invested £5460 partly in 4 per cent. stock at 96 and partly in 5 per cent. stock at $107\frac{1}{2}$ so that the income from each stock was the same. How much of each stock did he buy and what was his income after paying income tax at 4s. 9d. in the £? (L.Ch.C.)

29. £3585 is invested partly in $3\frac{1}{2}$ per cent. stock at 92 and partly in $4\frac{1}{4}$ per cent. stock at $97\frac{3}{4}$ so that the income derived from each stock is the same. Find (i) how much is invested in each stock, (ii) the total percentage yield, correct to two places of decimals.

30. A man who holds £3900 of 4 per cent. stock and £2400 of 7 per cent. stock determines to sell out, the former at 82 and the latter at 118, and invest the proceeds partly in a 5 per cent. stock at 96 and the rest in a 6 per cent. stock at 108, so as to get exactly the same income as before. How much of each stock must he buy? (R.S.A.)

31. A man sells £5833 6s. 8d. of $4\frac{1}{2}$ per cent. Consolidated Stock at 96 and invests the proceeds partly in 7 per cent. Railway Debentures at 135 and partly in $3\frac{1}{2}$ per cent. South African Stock at 75. The result is an increase in his income of £23 6s. 8d. Find how much of each stock he bought on reinvestment. (C.I.S.)

32. A man has an annual income of £550 from an investment in Railway stock paying $2\frac{3}{4}$ per cent. dividend. He increased his income to £600 by selling part of his Railway stock when it was quoted $76\frac{1}{8}$ – $77\frac{1}{8}$ and, with the proceeds, buying 5 per cent. War Loan when it was quoted $94\frac{7}{8}$ – $93\frac{7}{8}$. How much Railway stock did he sell, brokerage at $\frac{1}{8}$ per cent. being reckoned? (C.I.S.)

33. Two investments, one £7250 of $3\frac{1}{2}$ per cent. stock, and the other £5600 of $3\frac{3}{4}$ per cent. stock, are left to be shared amongst two men *A* and *B* such that their annual incomes are to be equal. Calculate how much stock each man must take and his annual income.

34. The capital of a company consists of 1,600,000 shares of £1 each, and it is now decided to increase the capital to 2,000,000 shares. Half of these new shares are offered at par to the present shareholders, each holder of 8 shares being offered one new share. The rest of the new shares are offered to the general public at £2 10s. per share. A man who already holds 200 shares accepts the new shares which are offered to him at par and is allotted an equal number of those offered at £2 10s. If the company pays a dividend of 12 per cent., what rate of interest will this man get on the new money he puts into the company? (R.S.A.)

35. The capital of a company consists of 600,000 £1 Preference Shares paying 6 per cent. and £24,000 of Ordinary Shares. There are also 3,000 Debentures of £100 each paying 7 per cent. Last year the Ordinary Shares paid 18 per cent. without altering the amount carried forward. This year it is expected that the net profits will be 10 per cent. greater than last year. If they are, and it is determined to put £5,000 to reserve, what rate of interest will be paid on the Ordinary Shares? (R.S.A.)

36. *B* holds twelve $4\frac{1}{2}$ per cent. debentures of £100 each, 800 $5\frac{1}{2}$ per cent. preference stock and a certain amount of ordinary stock in X.Y.Z. Ltd. On the 1st of June he receives a warrant for £60 7s. 6d., being a half-year's interest on the debentures and preference stock and an interim dividend of 7 per cent. on his holding of ordinary stock, all less tax at 5s. in the £. What was the nominal amount of *B*'s holding of ordinary stock? (C.I.S.)

37. The capital of a company consists of 485,000 8 per cent. Preference Shares of 10s. each and 1,200,000 Ordinary Shares at 1s. each. If the net profit for the year is £29,440 and £14,320 was

brought forward from last year and a dividend of $11\frac{1}{2}$ per cent. is declared on the Ordinary Shares, how much can be carried forward?
(R.S.A.)

38. The capital of a company consists of 1,200,000 ordinary shares of 5s. each and 200,000 $5\frac{1}{2}$ per cent. preference shares of £1 each. At what rate would dividend on the ordinary shares be payable if, at the end of a year, a profit of £47,000 was distributed among the shareholders?

A man held 500 ordinary shares and 80 preference shares. What should he receive after income tax at 4s. 9d. in the £ has been deducted?
(R.S.A.)

CHAPTER X

MENSURATION—MEASUREMENT OF AREA

10.1. Area.

EVERY plane figure has two dimensions, viz. lengths in two different directions, which are frequently called **length** and **breadth**. The measurements of these are always made in directions which are perpendicular to each other.

The measure of the space enclosed by the boundaries of a plane figure is called its **area**, and this may be found from the measurement of the length and breadth of the figure.

In the determination of area, since the directions of the dimensions involved are mutually perpendicular to each other, the **unit of area** is a **square** whose side is of **unit length**. Thus the area of a square whose side is one inch long is called a **square inch**; if the side is one foot long, the area is one **square foot**; when the side is one centimetre, the area is one **square centimetre**, and so on. Hence, the measurement of area really consists in finding how many unit squares are enclosed by any given figure.

10.2. Area of a Rectangle.

Let $ABCD$ (Fig. 3) be a rectangle drawn on a piece of squared paper ruled in squares each having a side of one-tenth of an inch. Each of these printed squares will be called a *small square*.

Suppose $AB = 2.3$ inches $= 23$ tenths of an inch, and $BC = 1.2$ inches $= 12$ tenths of an inch. Then the number of small squares enclosed by the rectangle $= 23 \times 12 = 276$.

Now in a square of one inch side, such as $AEFG$, the number of small squares $= 10 \times 10 = 100$;

∴ 100 small squares = 1 square inch,

so that 276 „ „ = $\frac{276}{100}$ sq. in. = 2·76 sq. in.

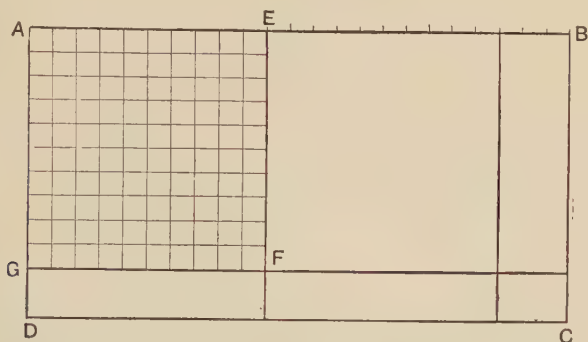


FIG. 3.—Area of a rectangle.

But this result could have been obtained at once, for

$$2\cdot3 \times 1\cdot2 = 2\cdot76 ;$$

i.e. the area may be found at once by multiplying the length AB by the breadth BC .

Hence, the area of a rectangle is measured by the product of the length and breadth, provided always the length and breadth are measured in the same units.

This important rule may be conveniently expressed symbolically, for if A , l , b denote the area, length and breadth respectively of a rectangle, then

$$A = l \times b.$$

In the case of a square whose side is s units long, then

$$l = b = s,$$

and the area becomes $s \times s = s^2$.

Therefore, if A be the area of a square of side s ,

$$A = s^2.$$

Ex. 1. Find the area of a rectangle whose sides are 23.5 in. and 17.6 in.

Evidently, from the above rule, the required area is

$$\begin{aligned} 23.5 \times 17.6 & \text{ square inches} \\ & = 413.6 \text{ sq. in.} \end{aligned}$$

10.3. Area or Square Measure.

It has already been seen from the square *AEFG* in Fig. 3 that one square inch contains $10 \times 10 = 100$ small squares, each of whose sides is $\frac{1}{10}$ inch. In the same way, a square foot contains

$$12^2 = 12 \times 12 = 144 \text{ square inches ;}$$

a square yard $3^2 = 9$ square feet ; a square metre $10^2 = 100$ square decimetres, and so on. This explains the following tables, which should be known thoroughly.

British Area Measure

12^2 or 144	sq. inches = 1 sq. foot
3^2 or 9	sq. feet = 1 sq. yard
$(5\frac{1}{2})^2$ or $30\frac{1}{4}$	sq. yards = 1 sq. pole
40	sq. poles = 1 rood
4	roods = 1 acre
640	acres = 1 sq. mile.

Since 22 yards = 1 chain ; also

$$\begin{aligned} 22^2 \text{ or } 484 & \text{ sq. yards} = 1 \text{ sq. chain,} \\ 10 & \text{ sq. chains} = 1 \text{ acre.} \end{aligned}$$

Note that the only new terms used are *rood* and *acre*, these being applicable only to area.

In the Metric system, the square metre is the unit used for measuring ordinary areas, whilst for land measurement the unit generally employed is the square dekametre, which is called an **are**. For large areas, the **hectare** is often used. Thus

$$\begin{aligned} 1 \text{ are} &= 1 \text{ square dekametre} = 100 \text{ square metres,} \\ 100 \text{ ares} &= 1 \text{ hectare,} \\ \text{and } 100 \text{ hectares} &= 1 \text{ square kilometre.} \end{aligned}$$

Ex. 2. *A local map is drawn to a scale of 16 chains to an inch ; how many square inches on the map will represent an area of 345·6 acres ?*

Since 1 inch represents 16 chains, then 1 square inch will represent 16×16 square chains.

But $345\cdot6$ acres $= 345\cdot6 \times 10$ sq. ch. $= 3456$ sq. ch., and this area will be represented on the map by

$$\frac{3456}{16 \times 16} \text{ square inches} = 13\cdot5 \text{ square inches.}$$

Ex. 3. *A man buys $1\frac{1}{2}$ acres of land at £1920 per acre and divides this land into rectangular plots 29 yd. 1 ft. long by $27\frac{1}{2}$ yd. wide. These he lets out on lease and charges the same ground rent for each, so that the total rent collected is $3\frac{1}{4}$ per cent. of the original cost. Calculate the annual ground rent he must charge for each plot.*

The area of each plot $= 29\frac{1}{3} \times 27\frac{1}{2}$ sq. yd.

$$\therefore \text{Number of plots} = \frac{1\frac{1}{2} \times 4840}{29\frac{1}{3} \times 27\frac{1}{2}} = \frac{3 \times 4840 \times 3 \times 2}{2 \times 88 \times 55} = 9.$$

Now, cost of land $= £(1920 \times 1\frac{1}{2}) = £2880$,

$$\text{and } 3\frac{1}{4}\% \text{ of this cost} = £\frac{3\frac{1}{4} \times 2880}{100} = £93\cdot6.$$

Hence, the nine plots have to yield, in ground rent, £93·6.

$$\therefore \text{Ground rent of each plot} = £\frac{93\cdot6}{9} = £10\cdot4 = £10 \text{ 8s.}$$

10·4. Some Important Practical Rules.

A few useful relations and rules, frequently employed in practice, will now be given.

1. *Relation between the British and Metric units of area.* Imagine a square of one inch side, then taking one inch to be equivalent to 2·54 centimetres, the length of each side of the square may be taken as 2·54 cm., and therefore its area in square centimetres is

$$2\cdot54 \times 2\cdot54 = 6\cdot4516.$$

Hence, to two places of decimals,

$$1 \text{ sq. in.} = 6.45 \text{ sq. cm.}$$

Similarly it may be shewn that

$$1 \text{ hectare} = 2.47 \text{ acres.}$$

2. *Measure of timber area.* In measuring the superficial area of wood for floors or match-lining, the builder generally uses a unit called a square, which is the area of a square of 10 feet side ;

$$\therefore \text{A square of timber} = 100 \text{ sq. ft.}$$

3. *Area of wall-paper.* Wall-paper is generally made in pieces 12 yd. long by 21 in. wide ; hence the area in square feet covered by each piece should be $36 \times 1\frac{3}{4} = 63$. Now in matching, etc., a certain area is bound to be wasted, and this is generally reckoned as *one-seventh*, so that the waste in each piece is 9 sq. ft., and therefore the actual area covered by a piece is $(63 - 9)$ sq. ft. or 54 sq. ft. Hence the following practical rule :

To find the number of pieces of wall-paper required for a room, divide the total area to be covered, in square feet, by 54.

10·5. Area of a Parallelogram.

Since the sides of a parallelogram are not, in general, at right angles to each other, the area will not be given by the product of

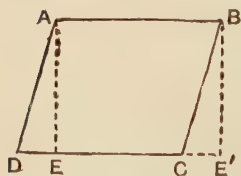


FIG. 4.

the lengths of adjacent sides as in the case of a rectangle. If, however, for any parallelogram $ABCD$ (Fig. 4), the triangle AED is cut off along AE , the perpendicular from A to DC , and then placed so that AD falls along BC , the parallelogram becomes the rectangle $ABE'E$, for ECE' is a straight line,

$$\angle BCE' = \angle ADE.$$

since

Hence, the area of $ABCD$ = area of rectangle $ABE'E$

$$= EE' \times E'B = DC \times EA ;$$

i.e. the area of a parallelogram is measured by the product of the length of one side and the perpendicular distance between that side and the opposite side.

10·6. Area of a Triangle.

Having found the area of a parallelogram, the area of a triangle may readily be deduced.

Let ABC (Fig. 5) be any triangle; draw AF perpendicular to BC , AE parallel to BC and CE parallel to BA meeting AE in E . Then $ABCE$ is a parallelogram of which AC is the diagonal. But the diagonal of a parallelogram bisects its area, so that, the area of the triangle ABC is half the area of the parallelogram $ABCE$.

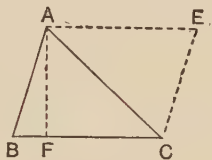


FIG. 5.

Hence, area of $ABC = \frac{1}{2} BC \times FA$.

Generally, the perpendicular FA is known as the altitude of the triangle, measured from the side BC ; hence, the area of a triangle is measured by half the product of one side and the altitude drawn to that side.

Since there are three sides, there are also three altitudes, and it is usual to denote the lengths of the sides BC , CA , AB by a , b , c respectively and the altitudes drawn to those sides as h_1 , h_2 , h_3 ; with this notation:

$$\text{area of triangle } ABC = \frac{1}{2}ah_1 = \frac{1}{2}bh_2 = \frac{1}{2}ch_3.$$

10·7. Area of a Trapezium.

A trapezium is a four-sided figure having one pair of opposite sides parallel. Let $ABCD$ (Fig. 6) be a trapezium having AD parallel to BC ; draw BF , CH perpendicular to AD and CE parallel to BA . Denote the lengths of the parallel sides AD , BC by a , b respectively and the distance, FB or HC , between them by h ; then

area of $ABCD$

$$= \text{area of parallelogram } AECB + \text{area of triangle } CED$$

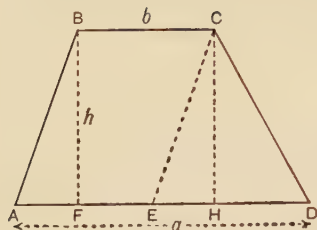


FIG. 6.

into which the trapezium may be divided by drawing either of the diagonals, AC or BD .

Ex. 4. *The plan of a piece of land, bounded by five straight lines PQ , QR , RS , ST , TP , is such that the line joining Q to S is parallel to PT and the line joining P to R is perpendicular to QS and intersects it at X . The following measurements are given on the plan :*

$$PT = 202 \text{ yd.}, \quad PX = 76 \text{ yd.}, \quad XR = 102 \text{ yd.}, \quad QS = 376 \text{ yd.}$$

Calculate the area of the land represented in acres, taking 4840 sq. yd. to an acre.

The plan is shewn in Fig. 7, and from this it is evident that the shape of the land is made up of a triangle QRS and a trapezium $QSTP$. Hence, the area of $PQRST$ in square yards

$$\begin{aligned} &= \frac{1}{2} \cdot XR \times QS + \frac{1}{2} \cdot PX \times (PT + QS) \\ &= \frac{1}{2} \cdot 102 \times 376 + \frac{1}{2} \cdot 76 \times (202 + 376) \\ &= 51 \times 376 + 38 \times 578 \\ &= 19,176 + 21,964 = 41,140. \end{aligned}$$

\therefore Number of acres in 41,140 square yards

$$= \frac{41140}{4840} = \frac{4114}{484} = 8.5.$$

\therefore The area of the land represented = 8.5 acres.

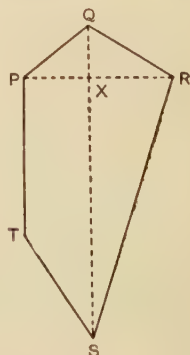


FIG. 7.

10-8. The Circle.

A circle is defined as the locus or path of a point which moves so that its distance from a given fixed point, called the **centre**, is always the same. This definition is well illustrated when a circle is described by an ordinary pair of compasses.

The curved boundary or path *APRQBS* (Fig. 8) traced out by the moving point is known as the **circumference**, and the constant distance *OA*, *OP*, *OQ* or *OB*, between any point on the circumference and the centre *O* is called the **radius**. Any straight line such as *PQ* or *AB* joining two points on the circumference is called a **chord**.

When a chord, like *AB*, also passes through the centre, it becomes a **diameter**; the diameter is thus twice the length of the radius, or denoting the radius and diameter of any circle by *r* and *d* respectively,

$$d = 2 \times r.$$

It has been established, both by accurate measurement and higher mathematics, that for every circle the ratio of circumference to diameter is constant, although it is impossible to express exactly the value of this constant. It is therefore usual to denote the ratio by the Greek letter π , pronounced *pi*; hence, if the length of the circumference be denoted by *c*,

$$\frac{c}{d} = \pi, \quad \text{or} \quad c = \pi \times d = 2 \times \pi \times r.$$

The value of the ratio denoted by π cannot be found either as an exact vulgar fraction or as a terminating decimal. Its value, to four places of decimals, is 3.1416, but for most purposes the approximate values 3.14 or $3\frac{1}{7}$ are generally sufficient.

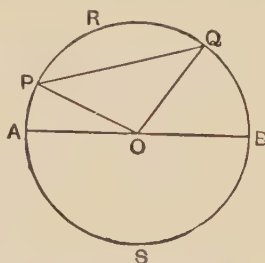


FIG. 8.—The circle.

Ex. 5. *At what rate, in miles per hour, is a cyclist travelling when each wheel, which is 26 inches in diameter, makes 126 revolutions per minute, taking $\pi = 3\frac{1}{7}$?*

Distance travelled in one revolution = Circumference of wheel

$$= 26 \times 3\frac{1}{2} \text{ in.}$$

$$\therefore \text{distance travelled in one hour} = 26 \times 3\frac{1}{2} \times 126 \times 60 \text{ in.}$$

$$= \frac{26 \times 3\frac{1}{2} \times 126 \times 60}{36 \times 1760} \text{ miles}$$

$$= \frac{26 \times 22 \times 126 \times 60}{36 \times 7 \times 1760} \text{ miles}$$

$$= \frac{13 \times 3}{4} \text{ miles} = 9\frac{3}{4} \text{ miles.}$$

Thus his rate is $9\frac{3}{4}$ miles per hour.

10·9. Area of a Circle.

When a figure is bounded by more than four straight sides it is known as a **polygon**; if the sides are all equal, the figure is said to be a **regular polygon**. From the area of a regular polygon it is quite simple to deduce the area of a circle.

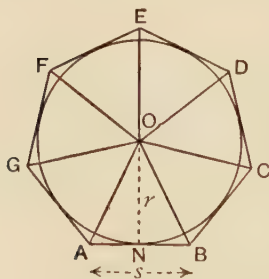


FIG. 9.

Let $ABCD \dots$ (Fig. 9) be a regular polygon of n sides circumscribed about a circle whose centre is O . Join each vertex to O and draw perpendiculars from O to each side of the polygon. These meet the sides at their points of contact with the circle, and are thus equal in length to the radius of the circle.

The polygon is divided in n equal triangles, OAB, OBC, \dots , all of which have equal bases, AB, BC, \dots , and equal heights.

Let r = radius of circle, and s = length of one of the equal sides of polygon; then

$$\text{area of polygon} = n \times (\text{area of one triangle } OAB)$$

$$\begin{aligned}
 &= n \times \frac{1}{2}(r \times s) = \frac{1}{2}r \times (n \times s) \\
 &= \frac{1}{2} \cdot r \times (\text{perimeter of polygon}).
 \end{aligned}$$

Now this is true however many sides the polygon may have, so that by increasing n the difference between the perimeter of the polygon and the circumference of the circle may be made very small; hence, when n is increased indefinitely, the area of the polygon becomes the area of the circle, i.e.

$$\begin{aligned}
 \text{area of circle} &= \frac{1}{2}r \times (\text{circumference}) \\
 &= r \times \frac{1}{2}(2 \times \pi \times r) = \pi \times r^2.
 \end{aligned}$$

Since $r = \frac{1}{2}d$, therefore $r^2 = \frac{1}{4}d^2$, so that $\pi \times r^2 = \frac{1}{4}\pi \times d^2$.

Hence, $\text{area of circle} = \pi \times (\text{radius})^2$ or $\frac{1}{4}\pi \times (\text{diameter})^2$.

Ex. 6. *The shape of a window is shewn in Fig. 10, $ABCD$ being a rectangle with a semicircular top AED . Find the cost of glazing the window at 4s. 6d. per square foot, if $AB = 8 \text{ ft. } 3 \text{ in.}$, $BC = 4 \text{ ft. } 8 \text{ in.}$ and $\pi = 3\frac{1}{7}$.*

Area of rectangle $ABCD$

$$\begin{aligned}
 &= (8\frac{1}{4} \times 4\frac{2}{3}) \text{ sq. ft.} \\
 &= \frac{33 \times 14}{4 \times 3} \text{ sq. ft.} = \frac{77}{2} \text{ sq. ft.} \\
 &= 38\frac{1}{2} \text{ sq. ft.}
 \end{aligned}$$

Radius of semicircular top

$$= \frac{1}{2} \cdot AD = 2\frac{1}{3} \text{ ft.}$$

\therefore area of semicircle AED

$$\begin{aligned}
 &= \frac{1}{2} \cdot \pi \times 2\frac{1}{3} \times 2\frac{1}{3} \text{ sq. ft.} \\
 &= \frac{22 \times 7 \times 7}{2 \times 7 \times 3 \times 3} \text{ sq. ft.} = \frac{77}{9} \text{ sq. ft.} \\
 &= 8\frac{5}{9} \text{ sq. ft.}
 \end{aligned}$$

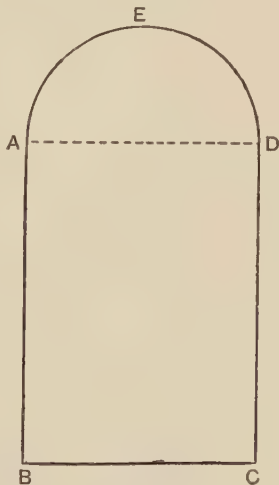


FIG. 10.

Hence, total area to be glazed

$$= (38\frac{1}{2} + 8\frac{5}{9}) \text{ sq. ft.} = 47\frac{1}{18} \text{ sq. ft.}$$

$$\begin{aligned}
 \therefore \text{Cost of glazing} &= \text{£} \frac{9}{40} \times 47 \frac{1}{8} \\
 &= \text{£} \frac{9 \times 847}{40 \times 18} = \text{£} \frac{847}{80} \\
 &= \text{£} 10 \text{ 11s. 9d.}
 \end{aligned}$$

To explain the method fully, the above working has been made longer than it need be. For examination purposes especially, the solution should be written out as follows :

$$\begin{aligned}
 \text{Total area in square feet} &= (8\frac{1}{4} \times 4\frac{2}{3}) + \frac{1}{2} \pi (2\frac{1}{3})^2 \\
 &= \frac{33 \times 14}{4 \times 3} + \frac{22 \times 7 \times 7}{2 \times 7 \times 3 \times 3} = \frac{77}{2} + \frac{77}{9} = \frac{77 \times 11}{18} . \\
 \therefore \text{Cost of glazing} &= \text{£} \frac{9 \times 77 \times 11}{40 \times 18} = \text{£} \frac{77 \times 11}{80} = \text{£} \frac{847}{80} \\
 &= \text{£} 10 \text{ 11s. 9d.}
 \end{aligned}$$

10-10. Area of a Circular Ring.

A circular ring or annulus is a figure bounded by two concentric circles, as shewn by Fig. 11. The area of the ring is clearly the difference between the areas of the two circles.

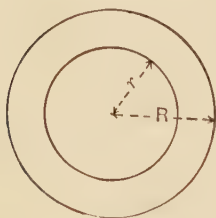


Fig. 11.

Let R , r be the radii of the outer and inner circles respectively, then

$$\text{area of larger circle} = \pi \times R^2,$$

$$\text{and area of smaller circle} = \pi \times r^2,$$

$$\therefore \text{area of ring} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2).$$

From algebra, it is known that

$$R^2 - r^2 = (R + r)(R - r),$$

so that the area of a circular ring $= \pi \times (R + r) \times (R - r)$,

i.e. the area of a circular ring $= \pi \times (\text{sum of radii}) \times (\text{difference of radii})$.

In practice, diameters are more conveniently measured ; hence, putting $D = 2R$ and $d = 2r$, the area of a circular ring becomes $\frac{1}{4} \pi (D + d)(D - d) = 0.7854(D + d)(D - d)$ taking $\pi = 3.1416$.

EXERCISES 10

Take (i) $\pi = 3\frac{1}{7}$ unless one of the other approximate values is given.

(ii) 1 acre = 4840 square yards.

1. A rectangular field is 277 yd. 2 ft. long and 90 yd. 2 ft. broad. Express its area (i) in square feet, (ii) in acres, correct to the nearest hundredth of an acre. (U.L.C.I.)

2. A rectangular plot of land 28 yards long and 33 feet broad is bought for £140. What is the price per acre? (R.S.A.)

3. The carpet, 1 foot 9 inches wide, required to cover a floor 19 feet 3 inches long by 16 feet wide, costs 15s. 9d. per yard. Find the number of yards bought and the total cost.

4. It is required to cover with linoleum the floor of a room in France measuring 12·7 metres by 7·62 metres. The linoleum is bought in London at 4s. 6d. per square yard. Taking 1 yard to be equivalent to 0·9144 metre, find the total cost.

5. What is the cost of flooring a room 14 ft. 3 in. long by 12 ft. 6 in. wide with boards which cost £1 12s. per square?

6. A carpet 13 ft. 2 in. long by 10 ft. 6 in. wide is sold for £13 16s. 6d. Find the price per square yard.

7. A rectangular piece of perforated zinc 5 ft. 4 in. long by 3 ft. 10½ in. wide costs 7s. 9d. Find the price per square foot.

8. Find the number of pieces of wall-paper required for a room 12 feet long, 10 feet 9 inches wide and 10 feet 6 inches high, allowing for a fireplace 5 ft. by 4 ft., a door 7 ft. by 3 ft. and a window 6 ft. 6 in. by 3 ft. 6 in.

9. A carpet is bought for a room 17 ft. long by 15 ft. 9 in. broad, so as to leave a margin of 18 inches all round. What will be the cost of the carpet at 24s. per square yard? (R.S.A.)

10. A local map is drawn to a scale of 16 chains to an inch. How many square inches on the map will represent an area of 345·6 acres?

11. A French map is drawn to the scale of 1 cm. to 1 km. and an English map of the same country to the scale of 1 inch to 1 mile. Find, to the hundredth of an inch, the distance on the English map between two places which are 7·4 cm. apart on the French map, taking 1 inch = 2·54 cm. (R.S.A.)

12. A surveyor's chain should measure 22 yards, but unknown to him, it has stretched $2\frac{3}{4}$ inches. He uses this chain to measure a field and calculate its area. How many square yards is each acre so calculated in error?

If he reckons that the area of the field is 8.73 acres, what is its true acreage? (R.S.A.)

13. A plan is drawn to the scale 3 chains to the inch. How many square inches on the plan will represent a field whose area is 15 acres 3 roods? (R.S.A.)

14. A border of linoleum one yard wide is placed round a room 20 feet by 17 feet. A carpet is placed in the middle of the floor so that it overlaps the linoleum to a width of six inches all round. Find the total cost if the carpet costs £1 1s. per square yard and the linoleum 4s. 6d. per square yard. (L.Ch.C.)

15. Calculate the number of pieces of wall-paper required for a room 22 ft. 6 in. long, 18 ft. wide and 12 ft. high, in which there are two windows each 6 ft. 6 in. by 4 ft. 6 in., a door 6 ft. by 3 ft. 9 in. and a fireplace 6 ft. by 4 ft. 6 in.

16. A man can buy 12 sheets of bromide paper, $12\frac{1}{2}$ in. by $10\frac{1}{2}$ in., for 6s. 11d., or 12 sheets of the same paper, $15\frac{1}{2}$ in. by $12\frac{1}{2}$ in. for 10s. 2d. Which is the cheaper rate and by how much in the £? (R.S.A.)

17. A plot of ground in the shape of a trapezium $ABCD$, having AB , DC as its parallel sides, is to be enclosed in order to contain 85.5 acres. On measurement, AB is 36 chains 83 links and CD has to be 34 chains 42 links in length. Calculate the perpendicular distance between AB and CD so that the required position of the boundary CD may be located.

18. An open space 360 acres in area is shewn on a plan by a four-sided figure $ABCD$ in which the diagonal BD is 13.5 cm. in length. The perpendiculars from A and C to BD are 4.9 cm. and 3.7 cm. in length respectively. Calculate the scale of the plan in inches per mile, taking 6.45 sq. cm. to one sq. in.

19. A plot of land is represented on a plan by a quadrilateral $ABCD$. AC is $18\frac{3}{4}$ in. long, the perpendiculars from B and D to AC are $7\frac{1}{2}$ in. and 8 in. respectively. If the scale of the plan is one inch to 4 chains, find the actual area of the land in acres.

20. A field whose area is $2\frac{1}{2}$ acres is divided into rectangular

plots 18 yd. 1 ft. long by 14 yd. 2 ft. wide. How many plots are there and what annual ground rent will they produce if all of them are leased at £7 19s. each per year?

21. The plan of a piece of ground is shewn as a six-sided figure $ABCDEF$. AD is vertical and the lines BGF , CHE are perpendicular to AD meeting it in G , H respectively. The measurements shewn are as follows :

$AG=90$ yd., $BG=47$ yd., $GF=63$ yd., $GH=86$ yd., $HC=109$ yd., $HE=133$ yd., and $HD=74$ yd.

Calculate the area of the ground in acres. (C.P.)

22. The plan of a plot of ground is a figure $ABCDE$ bounded by five straight lines, AB , BC , CD , DE , EA . The line joining A to C is parallel to DE and the line joining B to E intersects AC at right angles in F . The following measurements are shewn on the plan :

$AC=261$ yd., $DE=228$ yd., $BF=174$ yd., $FE=210$ yd.

Taking 4840 sq. yd. to an acre, calculate the acreage of the plot. (C.P.)

23. The plan of a farmland is shewn as a five-sided figure $ABCDE$; the sides AB , BC , CD , DE and EA being straight lines. CE is parallel to AB and DB meets CE at right angles in F . The actual measurements of the land are given as follows :

$AB=172$ yd., $FC=60$ yd., $DF=352$ yd., $FB=221$ yd., and $EF=296$ yd. Calculate the area of the farm in acres, taking 4840 square yards to an acre.

24. A bicycle wheel makes 700 revolutions in travelling one mile. How many revolutions does it make in travelling one kilometre, taking one inch to be equivalent to 2.540 cm.? Give the result correct to the nearest integer. (R.S.A.)

25. A bicycle wheel is 26 inches in diameter. At what rate, in miles per hour, is the bicycle travelling when this wheel is making 168 revolutions per minute? (R.S.A.)

26. In a circular plate nine inches in diameter, six holes each $1\frac{1}{2}$ inches in diameter are punched out. Find the area of the plate thus perforated to the nearest square inch, taking $\pi=3.14$.

27. On the same side of a line AB 4 ft. 8 in. long a quadrant ABC and a semicircle AEB are described. Find the area bounded by the two arcs AEB , AC and the straight line BC .

28. $ABCD$ is a square and AEC a circular arc whose centre is D and radius DA . The figure enclosed between AB , BC and the arc AEC is known as a *fillet*. Find the area, in square feet, of a fillet in which $AD = 2$ ft. 4 in.

29. A circular ring of metal has external and internal diameters of 26.9 in. and 17.9 in. respectively. Calculate the area of the ring in square feet.

30. A circular lake 321 feet in diameter is surrounded by a path eight feet wide. Find the cost of :

- (i) paving the path at 5s. 3d. per square yard,
- (ii) erecting a low fence round the boundary of the lake at 8s. 9d. per yard.

31. A park is represented on a plan by a four-sided figure $ABCD$ with a semicircle described externally on CD . To facilitate the measurements, a dotted line BF is shewn parallel to CD meeting DA in F , and another dotted line AGE is shewn perpendicular to DC meeting BF in G and DC in E . The measurements indicated are as follows :

$DC = 308$ yards, $BF = 132$ yards, $EG = 84$ yards and $GA = 126$ yards. Calculate the acreage of the park.

CHAPTER XI

THE CALCULATION OF SQUARE ROOT

11.1. The Square of a Number.

In Section 2.1, page 18, it is stated that a number raised to the second power is said to be **squared**; thus,

$$\begin{aligned}7 \text{ squared} &= 7^2 = 7 \times 7 = 49, \\59 \text{ squared} &= 59^2 = 59 \times 59 = 3481, \\32.7 \text{ squared} &= 32.7^2 = 32.7 \times 32.7 = 1069.29, \\ \frac{4}{5} \text{ squared} &= \left(\frac{4}{5}\right)^2 = \frac{4}{5} \times \frac{4}{5} = \frac{16}{25},\end{aligned}$$

and so on.

The process of squaring thus involves only simple multiplication.

11.2. The Reverse Process.

Each product obtained by squaring a number is called a **square number**; thus, 49, 3481, 1069.29, $\frac{16}{25}$, found above are square numbers. Now the reverse process of finding a number x which when squared is equal to a given number N is known as **finding the square root of N** ; this operation is represented symbolically by \sqrt{N} , so that, since $x^2 = N$, therefore $x = \sqrt{N}$.

Hence, from the examples above, $\sqrt{49} = 7$; $\sqrt{3481} = 59$; $\sqrt{1069.29} = 32.7$; $\sqrt{\frac{16}{25}} = \frac{4}{5}$.

It therefore becomes necessary to discover how the arithmetical square root of a number may be obtained.

11.3. The Area of a Square.

Let $ABCD$ (Fig. 12) be a square, H any point in AB and E a point in AD such that $AE = AH$. If HFK be drawn parallel to AD and EFG parallel to AB meeting HK in F , then $ABCD$ is divided into four smaller areas.

Denote the length of AH or AE by a and the length of HB or ED by b , then the area of

$$AHFE = AH \times EA = a \times a = a^2;$$

the area of

$$EFKD = EF \times DE = a \times b,$$

which is usually written ab , the sign of multiplication being omitted; the area of

$$FGCK = FG \times KF = b \times b = b^2;$$

the area of $HBGF = HB \times FH = b \times a = a \times b = ab$.

Hence, the area of the square $ABCD = \text{area of } AHFE + \text{area of } EFKD + \text{area of } FGCK + \text{area of } HBGF$

$$= a^2 + ab + b^2 + ab$$

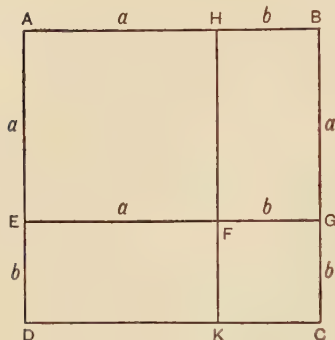
$$= a^2 + 2ab + b^2.$$


FIG. 12.

But the area of $ABCD = AB \times DA = (a+b) \times (a+b) = (a+b)^2$,

$$\therefore (a+b)^2 = a^2 + 2ab + b^2.$$

Note that the area of the square on AB is often written as AB^2 .

Those who have a knowledge of elementary algebra will be familiar with this important relation, which is generally called an **identity**. The method involved in finding the arithmetical square root of a number depends upon this algebraic identity, as will now be shown.

11·4. General Method of finding Arithmetical Square Root.

The identity $(a+b)^2 = a^2 + 2ab + b^2$ found in relation to the area of a square will now be applied in finding the square root of any number.

Ex. 1. Find the square root of 7921.

First mark off the digits in pairs from the unit; then to each pair there corresponds one digit of the square root.

Now suppose $79'21 = (a+b)^2 = a^2 + 2ab + b^2$.

To find a , the nearest square number less than 79 is obtained; this is obviously 64, which is the square of 8. Since the square root in this case contains two digits from the unit, $a=80$.

Hence, $7921 = 6400 + 2 \times 80 \times b + b^2$,

i.e. $2 \times 80 \times b + b^2 = 7921 - 6400 = 1521$,

or $b(160 + b) = 1521$.

To find b , note that the greatest integer in the quotient of $1521 \div 160$ is 9 and if $b=9$, $b(160 + b) = 9 \times 169 = 1521$, so that 9 is the necessary value of b .

$$\therefore \sqrt{7921} = 80 + 9 = 89.$$

The actual working is generally shown as follows :

	8 9	
	79'21	
	64 00	
$8 \times 2 = 16$	15 21	
169	15 21	$\therefore \text{Square Root} = 89.$

Ex. 2. *Extract the square root of 4719·69.*

	6 8· 7	
	47'19·69	Group the digits in pairs from the decimal
	36	point. Find the nearest square number smaller
128	11 19	than 47 ; this is 36 or 6^2 . Place the 6 above
	10 24	the 7 in 47, subtract 36 from 47 and bring
1367	95 69	down the next pair, viz. 19. Double the 6 just
	95 69	found, making 12, and divide this into 111 ;
		this gives 9, but $129 \times 9 = 1161$, which is too
		large, hence 8 must be tried. Now $128 \times 8 =$
		1024 , which is just smaller than 1119. Subtract, after placing the
		8 above the pair 19, and repeat the process, thus obtaining the last
		digit in the square root. In this way, $\sqrt{4719\cdot69} = 68\cdot7$.

Ex. 3. *A circular plate of area 301·84 square inches has to be cut from a sheet of thin metal. Taking $\pi = 3\frac{1}{7}$, calculate the diameter of the plate.*

If the diameter be d inches, then, from Section 10·9, the area of the plate in square inches is $\frac{1}{4}\pi d^2 = \frac{1}{4} \times \frac{22}{7} \times d^2 = \frac{11}{4} \times d^2$.

$$\therefore \frac{11}{4}d^2 = 301\cdot84,$$

from which $d^2 = \frac{301.84 \times 14}{11} = 27.44 \times 14 = 384.16.$

$$\therefore d = \sqrt{384.16}.$$

Working out this square root as shown on the right,

the required diameter = 19.6 inches.

	1 9. 6
	3'84.16
	1
29	2 84
	2 61
386	23 16
	23 16

EXERCISES 11A

Extract the square root of each of the following numbers :

1. 6241.

4. 2470.09.

7. $765\frac{4}{9}$.

2. 218089.

5. 3731.9881.

8. $8930\frac{1}{4}$.

3. 77316849.

6. 0.008649.

9. $653\frac{4}{49}$.

10. Find the diameter of a circular plate which must be cut so that its area is 346.5 square inches, taking $\pi = 3\frac{1}{7}$.

11. The area of a square field is 20.8849 hectares. Taking 1 hectare to be equal to one square hectometre, find the length of its side in metres and express this length also in yards, taking one yard to be equivalent to 0.914 metre.

12. Two fields have equal areas ; one is rectangular, being 578 ft. long and 242 ft. broad ; the other is square. Find the difference between the costs of fencing the fields at $4\frac{1}{4}$ d. per foot. (U.L.C.I.)

13. An area of 98.8 acres is represented on a map by a rectangle 1.9 inches by 1.3 inches. Find the scale, in inches to a mile, to which the map is drawn.

14. The diagonal d of a rectangular solid whose sides are a , b , c is given by the relation

$$d^2 = a^2 + b^2 + c^2.$$

Calculate d when $a = 84$ inches, $b = 44.8$ inches and $c = 56.1$ inches.

15. £2000 invested at compound interest for two years amounts to £2111 10s. 3d. Calculate the rate of interest per cent. per annum.

16. At the end of each year it is estimated that a motor car is only worth a fraction x of its value at the beginning of the year. Find the value of x as an ordinary fraction if a car costing £793 10s. becomes worth only £541 10s. at the end of two years.

17. A bought an article for £7 16s. 3d. and sold it to B at a profit of r per cent. on the cost price. B also sold it to C at the same profit per cent. on his cost price. If C paid £10 10s. 3d. for the article, find the value of r .

18. To allow for depreciation, a fixed percentage is deducted from the value of some machinery at the beginning of each year to estimate its value at the end of the year. If the machinery costs £740 and its value at the end of two years is estimated at £534 13s., calculate the percentage deduction made each year.

19. The amount of £560 when invested at compound interest for two years is £602 15s. 9d. Calculate the rate per cent. per annum of the interest added.

20. A certain sum put out at compound interest amounts to £4920 at the end of one year and to £5169 1s. 6d. at the end of three years. Calculate (i) the rate of interest per cent. per annum and (ii) the sum of money.

11.5. Approximate Square Root.

Actual square numbers are very few. Amongst the first hundred whole numbers, for instance, there are only *ten* square numbers, viz. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; the remaining ninety numbers are not, therefore, square numbers in so far that their square roots cannot be found exactly. By the general method, the square root may, however, be determined to any degree of approximation. Thus, to four places of decimals, $\sqrt{2} = 1.4142\dots$

But $(1.4142\dots)^2 = 1.99996164\dots$, which is less than 2 by 0.00003835...

To seven places of decimals, $\sqrt{2} = 1.4142135\dots$, and

$$(1.4142135\dots)^2 = 1.999999823\dots,$$

which is less than 2 by 0.000000176...

Such square roots which can only be found approximately are examples of what are known as **incommensurable numbers**, and most practical problems involve this kind of number; hence the necessity of determining to how many decimal places a result must be calculated in order to give a reliably accurate result.

Ex. 4. Calculate the square root of 43 correctly to eight places of decimals.

Since the result required is to be correct to eight places of decimals, the square root must be calculated to *nine* places.

	6· 5 5 7 4 3 8 5 2 4
	43·00 00 00 00 00 00 00 00 00
	36
125	7 00
	6 25
1305	75 00
	65 25
13107	9 75 00
	9 17 49
131144	57 51 00
	52 45 76
1311483	5 05 24 00
	3 93 44 49
13114868	1 11 79 51 00
	1 04 91 89 44
131148765	6 87 61 56 00
	6 55 74 38 25
1311487702	31 87 17 75 00
	26 22 97 54 04
13114877044	5 64 20 20 96 00
	5 24 59 50 81 76
	39 60 70 14 24

By the application of the general method, as shown above, it will be seen that the working gets more and more laborious, thus indicating that no exact number is likely to be found which when squared will be equal to 43 ; i.e. the square root of 43 is incommensurable, and to eight places of decimals, or to nine significant figures,

$$\sqrt{43} = 6.55743852.$$

It is only for special calculations that square roots are needed to more than three or four places, and in such cases the work may be slightly shortened by simple division after a few digits in the square root have been found by the general method.

In the present example, after determining the first five significant figures in the square root by the general method, the remaining five may be found by division ; thus,

$$\begin{array}{r}
 6 \cdot 5 \ 5 \ 7 \ 43852 \\
 \begin{array}{r}
 125 \overline{) 43 \cdot 00 \ 00 \ 00 \ 000000} \\
 \underline{36} \\
 7 \ 00 \\
 \underline{6 \ 25} \\
 1305 \overline{) 75 \ 00} \\
 \underline{65 \ 25} \\
 13107 \overline{) 9 \ 75 \ 00} \\
 \underline{9 \ 17 \ 49} \\
 131144 \overline{) 57 \ 51 \ 00} \\
 \underline{52 \ 45 \ 76} \\
 131148 \overline{) 5 \ 05 \ 240} \\
 \underline{3 \ 93 \ 444} \\
 1 \ 11 \ 7960 \\
 \underline{1 \ 04 \ 9184} \\
 6 \ 87760 \\
 \underline{6 \ 55740} \\
 320200 \\
 \underline{262296} \\
 57904
 \end{array}
 \end{array}$$

This is a repetition of the previous working.

The simple division begins here, the divisor being twice the square root already found, i.e. 65574×2 , omitting the decimal point.

Note that the last four digits obtained agree with those found previously by the general method.

$$\therefore \sqrt{43} = 6 \cdot 55743852.$$

Generally, the number of digits obtained by division will only be correct when it is one less than the number found by the full method ; hence the following rule :

When n digits of a square root have been obtained by the ordinary method, a further $(n - 1)$ digits may be found by dividing the last remainder by twice the square root already found.

11-6. Square Root of an Ordinary Fraction.

In some problems it may be convenient to use ordinary fractions, and occasionally the square root of such a fraction may be required. The following example will shew the alternative methods of doing this.

Ex. 5. Determine, as decimals to four places,

$$(i) \sqrt{\frac{53}{67}}, \quad (ii) \sqrt{\frac{37}{153}}.$$

(i) *First Method.*

As the square root is required in decimal form to four places, the given fraction may first be converted into a decimal by division. Ten places will be necessary so that the square root to five places may be obtained, and the final result correct to four places determined.

$$\text{Now } \frac{53}{67} = 0.7910447761 \dots ;$$

$$\therefore \sqrt{\frac{53}{67}} = \sqrt{0.7910447761 \dots} = 0.88940 \dots ,$$

working out the square root by the general method.

$$\text{Hence, correct to four places, } \sqrt{\frac{53}{67}} = 0.8894.$$

Second Method.

It is often convenient to convert the denominator into a square number, then only the square root of the numerator need be calculated. Since 67 is a prime number, the simplest way to convert the denominator into a square number is to multiply numerator and denominator by 67; thus :

$$\frac{53}{67} = \frac{53 \times 67}{67 \times 67} = \frac{3551}{67^2},$$

$$\therefore \sqrt{\frac{53}{67}} = \sqrt{\frac{3551}{67^2}} = \frac{\sqrt{3551}}{67} = \frac{59.59026 \dots}{67} = 0.88940 \dots ,$$

so that, correct to four places,

$$\sqrt{\frac{53}{67}} = 0.8894.$$

(ii) This may be evaluated by the first method under (i), but only the second method need be used here. To convert the denominator into a square number, note that $153 = 9 \times 17$, and 9

is a square number. Hence, it is only necessary to multiply numerator and denominator by 17; thus:

$$\frac{37}{153} = \frac{37 \times 17}{9 \times 17^2} = \frac{629}{3^2 \times 17^2},$$

$$\therefore \sqrt{\frac{37}{153}} = \sqrt{\frac{629}{3^2 \times 17^2}} = \frac{\sqrt{629}}{3 \times 17} = \frac{25.07987...}{51} = 0.49176... .$$

Hence, correct to four places of decimals,

$$\sqrt{\frac{37}{153}} = 0.4918.$$

11.7. The Right-angled Triangle.

One of the many applications of squares and square roots is connected with the mensuration of a right-angled triangle.

Let ABC (Fig. 13) be a triangle having a right angle at C ; denote the lengths of the sides BC , CA , AB by a , b , c respectively, the small letters corresponding to the lettering in capitals of the opposite angles. It may be shewn* that

$$AB^2 = BC^2 + CA^2,$$

or
$$c^2 = a^2 + b^2.$$

The longest side of a right-angled triangle is opposite the right angle and is called the *hypotenuse*, a word derived from the Greek meaning *a subtending line*, i.e. the side opposite the right angle.

Hence, in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This famous theorem was discovered by Pythagoras, a Greek mathematician who lived in Sicily, 570-500 B.C. It is therefore called the **Theorem of Pythagoras**, and its applications are very important.

* See the Author's *Mathematics for Technical Students* (Macmillan), pp. 183-185.

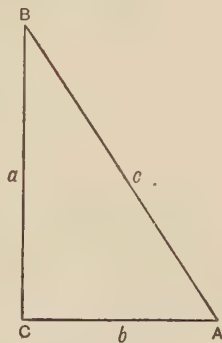


FIG. 13.

Ex. 6. *A rectangular field is 171 yards long and 157 yards wide ; what is the length of a path that runs diagonally across the field ?*

Let $ABCD$ (Fig. 14) represent the field, having $AB=CD=171$ yd. and $AD=BC=157$ yd. Suppose

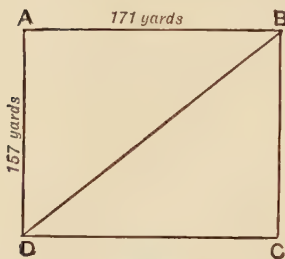


FIG. 14.

BD to be the path, then, since the triangle ABD is right-angled at A ,
 \therefore by the theorem of Pythagoras,

$$\begin{aligned} BD^2 &= DA^2 + AB^2 = 157^2 + 171^2 \\ &= 24649 + 29241 = 53890. \end{aligned}$$

$$\text{Hence, } BD = \sqrt{53890} = 232\cdot14\dots$$

on extracting the square root by the ordinary method.

$$\therefore \text{Length of path} = 232\cdot14 \text{ yards.}$$

11·8. Finding a Side which is not the Hypotenuse

From Fig. 13,

$$c^2 = a^2 + b^2,$$

so that

$$b^2 = c^2 - a^2 = (c+a)(c-a),$$

Since from algebra, it is known generally that

$$x^2 - y^2 = (x+y)(x-y).$$

This important result may perhaps be best remembered in the following statement :

The difference of the squares of two numbers is equal to the product of the sum of the numbers and their difference.

Ex. 7. *ABC is a triangle having a right angle at C and CD is drawn perpendicular to AB meeting it in D . If $CA=48\cdot7$ cm., $AB=75\cdot2$ cm., calculate (i) the length of CA , (ii) the area of the triangle ABC and (iii) the length of CD .*

The triangle is shewn in Fig. 15.

(i) Since $\angle C = \text{a right angle}$,

$$AB^2 = BC^2 + CA^2,$$

$$\therefore BC^2 = AB^2 - CA^2 = (75\cdot2)^2 - (48\cdot7)^2$$

$$= (75.2 + 48.7)(75.2 - 48.7) = 123.9 \times 26.5 = 3283.35.$$

Otherwise, by the longer method, $(75.2)^2 - (48.7)^2$

$$= 5655.04 - 2371.69 = 3283.35.$$

$$\therefore BC = \sqrt{3283.35} = 57.300\dots$$

Hence

the length of $BC = 57.30$ cm.

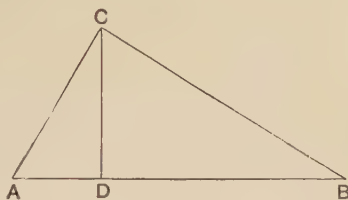


FIG. 15.

(ii) The area of triangle ABC ,

in square centimetres, $= \frac{1}{2} \cdot BC \times CA = \frac{1}{2} \times 57.3 \times 48.7 = 1395.255$,
i.e., correct to two places of decimals, area = **1395.26** sq. cm.

(iii) To find the length of CD , note that the area of the triangle in sq. cm. is also equal to $\frac{1}{2} \cdot CD \times AB = 37.6 \times CD$.

$$\therefore 37.6 \times CD = 1395.255,$$

from which

$$CD = \frac{1395.255}{37.6} = 37.107\dots;$$

$$\therefore \text{length of } CD = 37.1 \text{ cm.}$$

EXERCISES 11B

Find the square root of each of the following numbers :

- 29, to five places of decimals.
- 5366.7346, to five significant figures.
- 56479.8375, to six significant figures.
- $\frac{417}{578}$, correct to two decimal places.
- $\frac{47}{33}$, correct to three decimal places.

Find the difference between :

- $5.6 + 3.3 + 15.6$ and $\sqrt{(5.6)^2 + (3.3)^2 + (15.6)^2}$.
- $\sqrt{8.6 + 6.4 + 4.9}$ and $\sqrt{8.6} + \sqrt{6.4} + \sqrt{4.9}$ to two places of decimals.
- $\sqrt{6.37} - \sqrt{4.56}$ and $\sqrt{6.37 - 4.56}$ to three places of decimals.
- $\sqrt{(\frac{4}{7})^2 + (\frac{7}{9})^2}$ and $\frac{4}{7} + \frac{7}{9}$, to two places of decimals.

10. Calculate, to three places of decimals, the value of

$$\sqrt{78.45 \times 1.724} \div \sqrt{4369.21}.$$

11. A square map is designed to shew all places within 50 miles of London, and that area of the map which shews places beyond that distance is 134 square inches, to the nearest square inch. What is the scale of the map in miles per inch? (R.S.A.)

12. On a map a field of ten acres measures $9\frac{3}{4}$ square inches very nearly. Find the scale of the map, i.e. how many inches to the mile. (R.S.A.)

13. Evaluate the difference between $1\frac{6}{37}$ and $\sqrt{1.351}$, correctly to six places of decimals.

14. The population of a certain town in 1936 was 378,225 and in 1938 it became 380,816. Assuming that the increase per thousand to be approximately the same for each year, calculate the annual increase per thousand to one place of decimals.

(Note that the population increases on the compound interest principle.)

15. A manufacturer sells goods to a retailer at a profit of r per cent. on his selling price, and the retailer also sells the goods to the public at a profit of r per cent. on his selling price. If the cost of production of the goods is £327 and the price to the public is £507, calculate the value of r , correct to one place of decimals.

16. To how many places of decimals does the square root of 9.87 agree with π , taking $\pi = 3.1416$?

17. Find the radius of a circle whose area is to be 271.61 square feet, taking $\pi = 3.14$.

18. An ornamental park has the shape of a square with semi-circles described externally on each of its sides. The area of the park is $78\frac{3}{4}$ acres. Find (i) the side of the square in yards, (ii) the cost of fencing the park at 1s. 9d. per yard, taking $\pi = 3\frac{1}{7}$.

19. Shew that $\sqrt{11}$ lies between $3\frac{379}{1197}$ and $3\frac{120}{379}$. To how many places of decimals do the three results agree?

20. A man's income for each year is a fixed percentage of the income he received for the previous year. In 1936 his income was £578 per annum, and in 1938 it was £630 per annum. Calculate, to the nearest tenth, what the fixed percentage increase was.

21. To how many places of decimals is $\sqrt{1.53}$ the same as $\frac{47}{38}$?

(R.S.A.)

22. Two roads PR , QR intersect at right angles at R . A new straight road is cut from P to Q . If $PR=728$ yards and $QR=615$ yards, calculate how much shorter the distance is between P and Q by the new road PQ .

23. A triangular plot ABC has $AB=605$ yards, $CA=912$ yards and $\angle ABC=90^\circ$. Calculate (i) the length of BC (ii) the area of the plot in acres.

24. In a stairway $2\frac{1}{2}$ feet wide there are 15 steps. Each step rises 7 inches and has a tread of 11 inches. Find, to the nearest tenth of a square foot, the area of the sloping ceiling beneath the stairway. (L.Ch.C.)

25. The plan of a farmland consists of a five-sided figure $ABCDE$, the sides AB , BC , CD , DE , EA being straight lines. The line joining B to E is parallel to CD and the line joining A to C intersects EB at right angles in F . The measurements shewn are as follows :

$$CD=298 \text{ yd.}, EA=377 \text{ yd.}, EF=352 \text{ yd.}, FB=55 \text{ yd.},$$

$$FC=231 \text{ yd.}$$

Calculate (i) the length of AF and (ii) the area of the farm in acres.

26. The compound interest on £484 in two years is £22 0s. 6d. Calculate the rate of interest per cent. per annum.

27. The compound interest on a sum of money is £26 16s. 3d. for the first year and £28 11s. 8d. for the third year, these being calculated to the nearest penny. Find (i) the rate per cent. per annum at which the interest is added, and (ii) the principal.

28. A seven-acre field in the form of a rectangle has the lengths of its sides in the ratio of 2 : 3. It is desired to enclose it by a fence consisting of five parallel wires surrounding it. Calculate, to the nearest tenth of a yard, the length of wire required.

CHAPTER XII

THE MEASUREMENT OF VOLUME. DENSITY

12.1. Cubical Content.

IN the case of a solid body, not only length and breadth have to be considered, but also the thickness; thus every solid has **three dimensions**, which, for convenience, are generally measured in three mutually perpendicular directions.

The space occupied by a solid body is known as its **volume** or **cubical content**, and the unit of volume is the space occupied by a solid—called a **cube**—whose adjacent faces are perpendicular to each other and whose edges are all of the same length. If the edge is one inch in length, the unit is known as a **cubic inch**; if the length is one foot, the unit is one **cubic foot**, and so on. The size of a cubic centimetre is illustrated in Fig. 2, page 43.

The determination of volume therefore involves the finding of the number of appropriate unit cubes in a given solid.

12.2. Volume of a Rectangular Solid.

A solid bounded by three pairs of equal, parallel and rectangular faces is called a rectangular solid. Such a solid is shewn in Fig. 16;

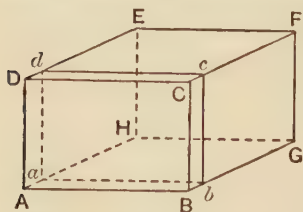


FIG. 16.—Volume of a rectangular solid.

the faces $ABCD$, $EFGH$ are equal, parallel and rectangular. So also are the faces $ABGH$, $CDEF$, and the faces $BCFG$, $ADEH$. Further, since each face is a rectangle, the three pairs are mutually perpendicular to each other. The length of any edge may be taken as the **length** of the solid, and the lengths of the two

perpendicular edges are then called the **breadth** and **height** of the solid

respectively. Thus in Fig. 16, CF may be taken as the length, CD as the breadth and BC as the height. The cube is thus a particular form of a rectangular solid, and for this reason, when length, breadth and height are not equal, the solid is often called a cuboid.

To find the volume of a rectangular solid or cuboid $ABCDEFGH$, let the length AH , the breadth AB and the height AD contain l , b and h units respectively.

Now consider a slice $ABCDdcba$ of unit thickness Aa ; then the number of unit cubes contained in this slice is equal to the number of unit squares in the rectangle $ABCD$, i.e. bh units of area.

\therefore the slice contains bh units of volume.

Hence the number of units of volume in the whole solid

$$\begin{aligned} &= bh \times \text{number of slices of unit thickness} \\ &= bh \times \text{number of units of length in } AH \\ &= bh \times l \text{ or } bhl. \end{aligned}$$

Therefore,

$$\text{Volume of cuboid} = bhl = \text{area of end} \times \text{length}.$$

For a cube, $b = h = l$,

$$\therefore \text{Volume of a cube} = l \times l \times l = l^3,$$

where l = length of one edge.

12.3. British and Metric Measures of Volume.

From the rule just established, it readily follows that the volume of a cube each of whose edges is one foot or 12 inches long, i.e. of a cubic foot, is, in cubic inches,

$$12^3 = 12 \times 12 \times 12 = 1728.$$

Exactly the same reasoning will shew that a cubic yard contains 3^3 or 27 cubic feet. Hence the following tables of cubic measure :

British Measure of Volume.

$$\begin{aligned} 12^3 \text{ or } 1728 \text{ cubic inches} &= 1 \text{ cubic foot,} \\ 3^3 \text{ or } 27 \text{ cubic feet} &= 1 \text{ cubic yard.} \end{aligned}$$

The Imperial Measure of Capacity for liquids and dry goods has already been given in Section 3·5 (page 40). The Imperial gallon is defined as that volume of distilled water which weighs 10 lb. This volume is 277·274 cubic inches, but an approximation often used is 277·25 cubic inches.

In practice, the weight of a cubic foot of water is frequently taken as 1000 ounces or 62·5 lb., so that, since a gallon weighs 10 lb., a cubic foot of water is equivalent to 6·25 gallons approximately.

In the Metric System, it will be clear, for reasons similar to those given earlier in this Section, that a cube, each of whose edges is one centimetre in length, contains 10^3 or 1000 cubic millimetres. Hence the following table :

Metric Measure of Volume.

0^3 or 1000 cubic millimetres	= 1 cubic centimetre (c.c.),
10^3 or 1000 „ centimetres	= 1 cubic decimetre,
10^3 or 1000 „ decimetres	= 1 cubic metre.

A cubic decimetre is called a **litre**, so that

$$1 \text{ litre} = 1000 \text{ cubic centimetres.}^*$$

A cubic metre is sometimes called a **stere**.

Ex. 1. *Taking one inch to be equivalent to 2·54 cm., find the number of cubic centimetres equivalent to a cubic inch. Use the result to determine the equivalent of a pint in litres, correct to three places of decimals, taking 1 gallon as 277·274 cub. in.*

Here we have first to find the volume of an inch cube in c.c.

Hence, since 1 in. = 2·54 cm.,

$\therefore 1 \text{ cub. in.} = 2·54 \times 2·54 \times 2·54 \text{ c.c.} = 16·387 \text{ c.c.,}$ correctly to three places of decimals.

$\therefore 1 \text{ cub. in. is equivalent to } 16·387 \text{ c.c.}$

Now $1 \text{ gallon} = 277·274 \text{ cub. in.,}$

* In accurate scientific work, the litre is taken as the unit and this is divided into 1000 millilitres.

so that

$$\begin{aligned} 1 \text{ pint} &= 277 \cdot 274 \div 8 = 34 \cdot 659 \text{ cub. in.} \\ &= 34 \cdot 659 \times 16 \cdot 387 \text{ c.c.} \\ &= 567 \cdot 80 \text{ c.c.,} \end{aligned}$$

i.e.

$$1 \text{ pint} = 568 \text{ c.c. to the nearest c.c.,}$$

and since 1000 c.c. = 1 litre ;

$$\therefore 1 \text{ pint is equivalent to } 0 \cdot 568 \text{ litre,}$$

and

$$1 \text{ litre is equivalent to } \frac{1}{0 \cdot 568} \text{ pints or } 1 \cdot 76 \text{ pints.}$$

Ex. 2. Calculate the freight for shipping abroad 33 rectangular packing cases each 4 ft. 9 in. by 2 ft. 8 in. by 1 ft. 3 in. at 28s. per load of 40 cubic feet.

The volume, in cubic feet, of each case = $4\frac{3}{4} \times 2\frac{2}{3} \times 1\frac{1}{4}$.

$$\therefore \text{Volume of 33 cases} = 4\frac{3}{4} \times 2\frac{2}{3} \times 1\frac{1}{4} \times 33 \text{ cu. ft.,}$$

$$\text{so that the number of loads} = \frac{4\frac{3}{4} \times 2\frac{2}{3} \times 1\frac{1}{4} \times 33}{40},$$

$$\text{and the freight} = \text{£} \frac{4\frac{3}{4} \times 2\frac{2}{3} \times 1\frac{1}{4} \times 33 \times 1\frac{2}{5}}{40}$$

$$= \text{£} \frac{19 \times 8 \times 5 \times 33 \times 7}{4 \times 3 \times 4 \times 40 \times 5} = \text{£} \frac{19 \times 11 \times 7}{4 \times 4 \times 5} = \text{£} \frac{1463}{80} = \text{£} 18 \text{ 5s. 9d.}$$

12·4. Some Practical Units of Volume.

In certain industries, special units of volume are adopted for convenience. The chief of these are as follows :

(i) In measuring timber, 1 load or ton is sometimes taken as 50 cubic feet of smooth timber

and a Petrograd standard = 165 cubic feet.

(ii) In measuring brickwork, it is usual to take an imaginary wall 16·5 ft. long, 16·5 ft. high and 13·5 in. thick as a standard rod.

Now the volume of this imaginary wall, in cubic feet,

$$= 16 \cdot 5 \times 16 \cdot 5 \times 13 \cdot 5 \div 12 = 306 \cdot 3 \dots$$

Hence, the standard rod is 306·3 cubic feet, but in practice, a standard rod of brickwork is generally taken as 306 cubic feet.

12.5. Solids of Uniform Section.

A solid whose section, cut perpendicular to its length, is always the same both in shape and size is called a **solid of uniform section**. When the end faces are also perpendicular to the length, the solid is called a **right solid of uniform section**.

Let $ABCD$ (Fig. 17) be any right solid of uniform section, and suppose a slice $AadD$ of unit thickness be cut off ; then the number

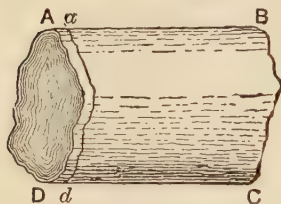


FIG. 17.—Solid of uniform section.

of unit cubes contained in this slice will be equal to the number of units of area in the face AD , i.e. in the section. Even if the face does not contain a whole number of units of area, any fraction of a unit will be the same fraction of a unit cube, since the whole slice is of unit thickness. Hence, in every case, the area

of the section is a numerical measure of the volume of the slice. The whole solid can be cut into as many slices of unit thickness as there are units in its length ;

∴ Volume of a right solid of uniform section is measured by the product area of section \times length of solid.

12.6. Prisms and Cylinders.

When the end faces of a solid of uniform section are bounded by straight lines, the solid is usually called a **prism** ; when they are bounded by curved lines the solid is called a **cylinder**. When the end faces are perpendicular to the length the solid is called a **right prism** or a **right cylinder**. When the faces are parallel but not perpendicular to the length, the solid is known as an **oblique prism** or an **oblique cylinder** as the case may be.

The shape of the section of a prism gives it its name ; thus when the section is a triangle, the solid is a **triangular prism**, when a hexagon, the solid is a **hexagonal prism**, and so on.

The cylinder with circular section is so important in practice that it is useful to express its volume in symbols.

Let R be the radius of the circular section of a right cylinder whose length is l , R and l being measured in the same units, then,

$$\text{Volume of right circular cylinder} = \pi R^2 l.$$

If a cylindrical hole of radius r be bored axially through the solid cylinder, it becomes what is known as a **hollow right cylinder**. Now the volume of the hole is that of a cylinder of length l and radius r , so that the volume of a hollow right cylinder is

$$\pi R^2 l - \pi r^2 l = \pi l (R^2 - r^2) = \pi l (R + r)(R - r).$$

Alternatively, the volume may be found from the general formula for a solid of uniform section, for its section is a ring whose external and internal radii are R and r respectively; hence the sectional area is $\pi(R + r)(R - r)$, by Section 10·10. Therefore

the volume of a hollow right cylinder of length l whose external and internal radii are R , r respectively $= \pi l (R + r)(R - r)$.

Ex. 3. Find the capacity, in litres to the nearest tenth, of a measuring glass in the form of a hollow right cylinder of diameter 12·6 cm. and height 20·1 cm., taking $\pi = 3\frac{1}{7}$.

Radius of cylindrical vessel = 6·3 cm.

\therefore Volume $= \pi \times (6\cdot3)^2 \times 20\cdot1$ c.c.

$$= \frac{22 \times 6\cdot3 \times 6\cdot3 \times 20\cdot1}{7 \times 1000} \text{ litres, since 1 litre} = 1000 \text{ c.c.}$$

$$= \frac{2507\cdot274}{1000} \text{ litres} = 2\cdot507... \text{ litres.}$$

\therefore Required capacity = 2·5 litres.

Ex. 4. Tar is contained in cylindrical barrels of diameter 1 ft. 10 in. and height 2 ft. 7½ in. Find the number of such barrels required to cover with tar, to an average depth of $\frac{1}{8}$ inch, the surface of a road 1 mile 418 yards long and 22 feet wide. Take $\pi = 3\frac{1}{7}$.

Volume of tar in each barrel $= \pi \times (11)^2 \times 31\frac{1}{2}$ cu. in.

Volume of tar required for the surface of the road

$$= (1760 + 418) \times 36 \times 22 \times 12 \times \frac{1}{8} \text{ cu. in.}$$

$$= 2178 \times 36 \times 22 \times 12 \times \frac{1}{8} \text{ cu. in.}$$

\therefore Number of barrels of tar required

$$= \frac{2178 \times 36 \times 22 \times 12 \times \frac{1}{8}}{\pi \times (11)^2 \times 31\frac{1}{2}}$$

$$= \frac{2178 \times 36 \times 22 \times 12 \times 7 \times 2}{22 \times 121 \times 63 \times 8} = 18 \times 12 = 216.$$

12.7. Surface Area of a Solid.

The total area of all the external faces of a solid is called its **surface area**. If, however, the areas of the end faces are not included, the total area of the remaining faces is known as the **lateral area**. When the faces are plane, they are usually simple rectilinear figures whose areas can be easily calculated.

Ex. 5. Find the cost of sheet zinc required to line the inside of an open rectangular tank 5 ft. 3 in. long, 4 ft. 6 in. wide and 2 ft. 7 in. deep, allowing 10 per cent. of the net area for overlapping and waste. at $10\frac{1}{2}$ d. per square foot.

Area, in square feet, of the four vertical sides

$$= 2(5\frac{1}{4} \times 2\frac{7}{2}) + 2(4\frac{1}{2} \times 2\frac{7}{2}) = 2 \times 2\frac{7}{2}(5\frac{1}{4} + 4\frac{1}{2}) = 5\frac{1}{6} \times 9\frac{3}{4} = \frac{403}{8}.$$

The bottom has also to be covered, and its area in square feet

$$= 5\frac{1}{4} \times 4\frac{1}{2} = \frac{189}{8}.$$

\therefore Total area, in square feet, to be covered

$$= \frac{403}{8} + \frac{189}{8} = \frac{592}{8} = 74.$$

Add 10% of this for overlapping and waste, then total area of sheet zinc required = $\frac{74 \times 11}{10}$ sq. ft.

Hence, since $10\frac{1}{2}\text{d.} = \frac{7}{8}$ of a shilling, the total cost

$$\frac{74 \times 11 \times 7}{10 \times 8} \text{ shillings} = \frac{2849}{40} \text{ shillings} = 71\frac{9}{40} \text{ shillings} \\ = \text{£}3 \text{ 11s. } 2\cdot7\text{d.}$$

\therefore to the nearest penny, the cost is **£3 11s. 3d.**

12·8. Case of a Solid Circular Cylinder.

The lateral area of a right circular cylinder may easily be determined. Suppose the curved surface to be covered completely with paper without overlapping, then, on opening the paper out into a plane sheet, it is seen to be rectangular in shape, the length being the same as that of the cylinder and the breadth being equal to the circumference of the circular section. If r = radius of section and l = length or height of cylinder, then its lateral area = $l \times 2\pi r = 2\pi rl$.

For the total surface, the areas of the circular ends must be included, so that the surface area of the cylinder = (lateral area) + (areas of end faces) = $2\pi rl + 2\pi r^2 = 2\pi r(l + r)$.

Hence, for a cylinder of length l and base radius r ,

$$\text{Lateral area} = 2\pi rl \quad \text{and} \quad \text{total surface area} = 2\pi r(l + r).$$

Ex. 6. *An ordinary tin consists of a hollow right cylinder open at the top with a detachable lid which is also of the form of a shallow hollow cylinder of the same diameter. Calculate the area, in square feet, of the sheet metal required to make one gross of such tins, with lids, if the diameter is to be 5 inches, the height $5\frac{1}{2}$ inches, and the depth of the lid $\frac{3}{4}$ inch, taking $\pi = 3\frac{1}{7}$.*

Area of metal, in square inches, required for each tin

$$= \text{area of curved side} + \text{area of bottom} \\ = 2\pi \cdot 2\frac{1}{2} \times 5\frac{1}{2} + \pi \cdot (2\frac{1}{2})^2 = \frac{605}{7} + \frac{275}{14} = \frac{1485}{14}.$$

Similarly, the area of metal required for the lid, in sq. in.,

$$= 2\pi \cdot 2\frac{1}{2} \times \frac{3}{4} + \pi \cdot (2\frac{1}{2})^2 = \frac{165}{14} + \frac{275}{14} = \frac{440}{14},$$

$$\therefore \text{total area in sq. in.} = \frac{1485}{14} + \frac{440}{14} = \frac{1925}{14} = \frac{275}{2}.$$

$$x = \frac{18.2 \times 1000}{100 \times 100 \times 11.4} = \frac{9.1}{57} = 0.159 \dots$$

\therefore Thickness in millimetres $= 0.159... \times 10 = 1.59...$
 $= 1.6$, to one decimal place.

Ex. 8. *The external diameter of a cast iron pipe of length one yard is 6 inches. It is uniformly $\frac{1}{4}$ in. thick. Find the weight of this pipe to the nearest lb., if 1 cubic inch of cast iron weighs 0.26 lb. and $\pi = 3.142$.* (U.L.C.I., 1938.)

This pipe is a right hollow cylinder of length 1 yard, or 36 inches, external radius 3 inches, and internal radius $(3 - \frac{1}{4})$ inches $= 2\frac{3}{4}$ inches.

Hence, by the formula of Section 12.6,

$$\begin{aligned}\text{volume of pipe} &= \pi \times 36 \times (3 + 2\frac{3}{4}) \times (3 - 2\frac{3}{4}) \text{ cu. in.} \\ &= 3.142 \times 36 \times 5.75 \times 0.25 \text{ cu. in. ;}\end{aligned}$$

and since 1 cubic inch of the material weighs 0.26 lb.,

$$\begin{aligned}\therefore \text{weight of pipe} &= 3.142 \times 36 \times 5.75 \times 0.25 \times 0.26 \text{ lb.} \\ &= 3.142 \times 9 \times 1.495 \text{ lb.} = 42.27561 \text{ lb.}\end{aligned}$$

Hence, to the nearest tenth of a lb.,

$$\text{Weight} = 42.3 \text{ lb.}$$

EXERCISES 12

Take (i) $\pi = 3\frac{1}{7}$ where no other value is stated,
 (ii) $6\frac{1}{4}$ gallons as equivalent to 1 cubic foot.

1. An open tank is made of wood $1\frac{1}{2}$ in. thick. How much water, in gallons, will it hold if the external dimensions are 8 ft. 3 in. long, 5 ft. 6 in. wide and 2 ft. deep? (U.L.C.I.)

2. Taking a standard rod of brickwork as the volume, in cubic feet, of a wall $16\frac{1}{2}$ ft. long, $16\frac{1}{2}$ ft. high and $13\frac{1}{2}$ in. thick, and the measurements of a brick to be 9 in. by $4\frac{1}{2}$ in. by 3 in., calculate the number of bricks to a standard rod.

3. A rectangular sheet of metal 1.35 metres long, 74 centimetres wide, $2\frac{1}{2}$ millimetres thick, weighs 18 kilograms. Calculate in grams, to one place of decimals, the weight of one cubic centimetre of the metal. (R.S.A.)

4. A rectangular tank on a square base, 3 ft. 6 in. deep, is to be constructed to hold 224 gallons of water. Calculate the required length of a side of the base.

5. A rectangular block of stone, weighing 158 lb. per cubic foot, has to be cut. It must have a weight of $118\frac{1}{2}$ lb., a length of 2 ft. 3 in. and a breadth of 1 ft. 3 in. Calculate the thickness of the block in inches.

6. Find, in kilograms to one decimal place, the weight of a slab of stone 74.6 cm. long, 37.6 cm. broad and 8.5 cm. thick, given that 1 c.c. of the stone weighs 2.7 grams. (R.S.A.)

7. The external dimensions of a packing case are 3 ft. by 2 ft. $5\frac{1}{2}$ in. by 1 ft. 8 in., and the wood is $\frac{3}{4}$ in. thick. If the wood weighs 82 lb. per cubic foot, find the weight of the packing case to the nearest ounce. (R.S.A.)

8. The bottom of a rectangular tank is 8 feet long and $4\frac{1}{2}$ feet wide. What will be the depth of the water when the tank contains 900 gallons?

9. A rectangular bar of steel 14 ft. long, 5 ft. wide and $1\frac{1}{2}$ in. thick, weighs $38\frac{1}{4}$ cwt. Find the weight of a bar of the same material one inch square in section and one yard long.

10. A rectangular tank, 10 ft. 8 in. long by 6 ft. 5 in. wide, is to be constructed to hold 1925 gallons. Find (i) the depth of the tank, and (ii) the area of iron plate required, to the nearest square foot, neglecting the thickness of the material and including a lid. (C.P.)

11. Two hollow boxes are to be made from a sheet of iron 60.75 square feet in area. One is to be a cuboid having its dimensions in the ratio of 10 : 6 : 3, and the other is to be a cube. The surface areas of the boxes are to be equal ; find the dimensions of each.

12. A block of stone 2 ft. 4 in. long, 1 ft. 10 in. broad and 6 in. thick weighs 320 lb. Find, in inches and an ordinary fraction of an inch, the thickness of another block of the same kind of stone which weighs 350 lb. and is 4 ft. 1 in. long and one foot broad. (R.S.A.)

13. A litre measure has the form of a hollow right cylinder, open at the top and closed at the bottom by a flat circular plate. Its full capacity is 1 litre and its inside diameter is 8.6 centimetres. Calculate its internal height in centimetres to one place of decimals. (R.S.A.)

14. It is required to construct a cylindrical tank on a circular base capable of containing 360 gallons of petrol. What must be the height of the tank, to the nearest 0.01 inch, if the internal diameter is to be 60 inches? Take $\pi = 3.1416$ and 1 gallon = 277.274 cubic inches. (R.S.A.)

15. A trench is dug in order to lay an electric cable 4 inches in diameter. When the trench is filled in again with the earth pressed down so as to be as compact as before, and the level the same as before, the loose earth left is carted away. If the volume of this loose earth is 10 per cent. greater than before it was excavated, find, to the nearest integer, how many cart-loads are taken away for each mile of trench, taking a cart-load = 1 cubic yard. (R.S.A.)

16. Water runs into a cylindrical tank standing upright at the rate of 90 gallons per minute, the inside horizontal diameter of the tank being 20 inches. Calculate, in inches per second to two places of decimals, the rate at which the surface of the water is rising in the tank. (R.S.A.)

17. A cylindrical pipe of 3 cm. bore, i.e. internal diameter, running full of water, is delivering 20 gallons per minute. At what rate, in kilometres per hour, is the water travelling along the pipe? Take $\pi = 3.142$ and 1 litre = 0.220 gallon. Give the result correct to one decimal place. (R.S.A.)

18. An iron bar 1 ft. 2 in. long has a uniform square section whose side is $5\frac{1}{2}$ inches. The bar is melted into a cylindrical rod 3 ft. 8 in. in length without loss of volume. Calculate the diameter of the rod. (C.P.)

19. Tar is contained in cylindrical vessels, diameter 1 ft. 10 in., height 2 ft. $7\frac{1}{2}$ in. How many of these are required in order to cover with tar to an average depth of $\frac{1}{8}$ inch the surface of a road 1 mile 248 yards long and 22 feet wide? (R.S.A.)

20. A rectangular metal plate of uniform thickness is 15.7 in. long and 8.9 in. broad. By what percentage, to two places of decimals, will its weight be reduced if two circular holes, each of diameter 5.8 in., are cut in the plate? (U.L.C.I.)

21. A cubical chest whose internal measurement is 2 ft. 3 in. each way is full of tea. The tea has to fill 462 equal cylindrical canisters each $6\frac{3}{4}$ inches in height. Calculate the diameter of each canister to two places of decimals.

22. A storage bin for fine sand is a rectangular box 8 ft. 9 in. long, 5 ft. 6 in. wide and 3 ft. 9 in. deep. The sand is tipped into the bin from cylindrical buckets, each 1 ft. 3 in. high and 10.5 in. in diameter. Calculate the number of bucketfuls of sand needed to fill the bin completely. (C.P.)

23. On a cubical pedestal a hollow circular cylinder of the same material and weight is to be placed. The internal and external diameters of the cylinder are to be 5.28 ft. and 7.28 ft. respectively, and the edge of the cube is 9.42 ft. Calculate the required height of the cylinder to the nearest tenth of a foot, taking $\pi = 3.14$.

24. An open cylindrical tank of diameter 12 feet is to be constructed to hold 2826 gallons of water. Find the area of sheet iron required, including the circular base. Take $\pi = 3.14$.

25. A circular fish pond $28\frac{1}{2}$ feet in diameter with a flat bottom is to be made in the centre of a square courtyard whose side is 38 yards long. To save the expense of removal, the earth excavated is spread evenly over the remaining surface of the courtyard. Find, in inches to the nearest tenth, how much the level of the courtyard will be raised when the bottom of the pond is $5\frac{1}{4}$ feet below the new level.

26. Calculate the cost of lining the inside of a closed cylindrical tank of internal diameter 4 ft. 6 in. and length 8 ft. 3 in. at 10d. per square foot.

27. Three cylindrical tins have heights 7 inches, 9 inches and 16 inches respectively. The respective diameters of the first two are 26 inches and 38 inches. The volume of the third tin is equal to the sum of the volumes of the other two; calculate its diameter to the nearest tenth of an inch.

28. A wooden curtain rod is a cylinder of length 7 feet and diameter 3 inches and it weighs 17 lb. 14 oz. Find the weight of one cubic foot of the wood. (R.S.A.)

29. If one cubic foot of iron weighs 435 lb., find to the nearest foot the length of iron rod $\frac{7}{8}$ inch in diameter which will weigh one cwt. (R.S.A.)

30. An ordinary washer is made by cutting a circular hole from the centre of a disc of metal. Find, in lb., the weight of a gross of washers 0.1 in. thick if the diameters of the disc and the hole are 1.9 in. and 0.5 in. respectively, given that a cubic foot of the metal weighs 450 lb. (C.P.)

31. A metal pipe of circular section has an outer diameter of $3\frac{3}{4}$ in. and the thickness of the metal is $\frac{1}{8}$ inch. If one cubic foot of the metal weighs 710 lb., find the weight of one yard length of the pipe to the nearest ounce. (R.S.A.)

32. A brass tube is 4 feet long, its external diameter is 2 inches, and its internal diameter is 1.9 inches. Given that one cubic foot of brass weighs 521 lb., find, to the nearest ounce, the weight of the tube. (R.S.A.)

33. Lead piping of circular section is of 2 cm. bore and 2 mm. thickness. Calculate in kilograms to three significant figures, the weight of a metre length of it, assuming that one cubic centimetre of lead weighs 11.4 grams. (R.S.A.)

34. A cylindrical tank holds 160 gallons of oil. If the internal height of the tank is 3 ft. 4 in., calculate to the nearest hundredth of a foot the internal diameter of the circular base. (U.L.C.I.)

35. A thread of mercury is drawn into a long glass tube. On measurement the thread is 10.5 cm. long, and its weight is 10.098 grams. Calculate, in millimetres, the bore of the glass tube, taking one cubic centimetre of mercury to weigh 13.6 grams.

36. The weight of a column of mercury of length 11 cm. in a glass tube is 785.4 grams. If one cubic centimetre of mercury weighs 13.6 grams, find the internal diameter of the tube in cm., correct to two places of decimals. (L.Ch.C.)

37. One cubic foot of copper weighs 540 lb. and one cubic foot of zinc weighs 437 lb. Find, to the nearest lb., the weight of a cubic foot of brass made by mixing 64 lb. of copper with 36 lb. of zinc. (R.S.A.)

38. Find, to the nearest tenth of a lb., the weight of a cubic foot of lead if one cubic centimetre weighs 11.35 grams, assuming that one cubic foot = 0.0283 cubic metre and 1 kilogram = 2.205 lb. (L.Ch.C.)

39. A food is sold in cylindrical tins of two sizes. One, measuring internally 15.3 cm. in height and 12.4 cm. in diameter, is sold for 2s. 6d. The other, measuring internally 18.7 cm. in height and 18.6 cm. in diameter, is sold for 5s. 6d. How much per cent. of the purchase price is the larger tin cheaper than the smaller?

40. A solid ornament consisting mainly of lead with an external coating of silver, weighs 1158 grams and has a volume of 103 c.c.

Taking the weight of 1 c.c. of lead to be 11·4 grams and the weight of 1 c.c. of silver to be 10·5 grams, calculate the value of the silver in the ornament at 32 centimes per gram.

41. A cylindrical tin, 7 inches high and 3 inches in diameter internally, is filled with a mixture of two powders, one of which weighs 62 lb. and the other 51 lb. per cubic foot. The empty tin weighs 2 oz. and the full tin weighs 1 lb. 12 oz. Find the weight of the heavier powder in the tin.
(R.S.A.)

PART II

CHAPTER XIII

INDICES AND LOGARITHMS

13.1. The Fundamental Laws of Indices.

THE index notation has already been briefly explained in Section 2.1, page 18. It only remains to consider the fundamental operations of indices in relation to their application to practical problems.

Let a represent any number, then

$$a^5 \times a^3 = (a \times a \times a \times a \times a) \times (a \times a \times a) = a^8.$$

In general, if m and n denote two numbers,

$$a^m \times a^n = a^{m+n}. \dots\dots\dots(i)$$

Thus in multiplying powers of the same number the indices are added.

$$\text{Again, } a^9 \div a^7 = \frac{a \times a \times a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a} = a \times a = a^2,$$

$$\text{i.e. } a^m \div a^n = a^{m-n}. \dots\dots\dots(ii)$$

Hence, in dividing powers of the same number, the index of the quotient is found by subtracting the index of the divisor from that of the dividend.

The relations established in (i) and (ii) constitute the fundamental laws of indices.

13.2. Some Special Cases.

For a power of a power apply (i);

$$\text{for example, } (a^5)^3 = a^5 \times a^5 \times a^5 = a^{5+5+5} = a^{15}.$$

$$\text{In general, } (a^m)^n = a^{mn}. \dots\dots\dots(iii)$$

Again,

$$a^m \div a^m = 1.$$

But, by (ii),

$$a^m \div a^m = a^{m-m} = a^0;$$

$$\therefore a^0 = 1. \dots\dots\dots(\text{iv})$$

Sometimes a problem may involve a negative index.

Now, if m represents a positive number; then

$$a^{-m} \times a^m = a^{-m+m}, \text{ by (i),}$$

$$= a^0 = 1, \text{ by (iv);}$$

\therefore dividing out by a^m ,

$$a^{-m} = \frac{1}{a^m}. \dots\dots\dots(\text{v})$$

A root may also be expressed in the index notation, for since

$$1 = \frac{1}{2} + \frac{1}{2}, \text{ or } \frac{1}{3} + \frac{1}{3} + \frac{1}{3}, \text{ or } \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}, \text{ or } \dots;$$

$$\therefore a^1 = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} \times a^{\frac{1}{4}} = \dots$$

$$\text{But } a = \sqrt{a} \times \sqrt{a} = \sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = \sqrt[4]{a} \times \sqrt[4]{a} \times \sqrt[4]{a} \times \sqrt[4]{a} = \dots$$

$$\therefore \sqrt{a} = a^{\frac{1}{2}}; \sqrt[3]{a} = a^{\frac{1}{3}}; \sqrt[4]{a} = a^{\frac{1}{4}}; \text{ and so on.}$$

$$\text{In general, } \sqrt[n]{a} = a^{\frac{1}{n}}. \dots\dots\dots(\text{vi})$$

$$\text{Finally, since } (a^n)^m = a^{\frac{m}{n}} \text{ by (iii), and } a^{\frac{1}{n}} = \sqrt[n]{a},$$

$$\therefore a^{\frac{m}{n}} = \sqrt[n]{a^m}. \dots\dots\dots(\text{vii})$$

13.3. Logarithms.

Since multiplication and division of powers of the same number can conveniently be carried out by addition and subtraction of indices, a simple practical method of calculating is possible if every number can be expressed as a power of some standard number, called the **base**. The common base chosen is ten. By higher mathematics, it has been calculated that :

$$2 = 10^{0.3010},$$

$$7 = 10^{0.8451},$$

$$11 = 10^{1.0414},$$

$$3 = 10^{0.4771},$$

$$8 = 10^{0.9031},$$

$$13 = 10^{1.1139},$$

$$5 = 10^{0.6990},$$

$$9 = 10^{0.9542},$$

$$15 = 10^{1.1761},$$

and so on for all numbers, only a few of which have been given. It should be observed that since 11, 13, 15 lie between 10 and 100, i.e. between 10^1 and 10^2 , the power of 10 representing each of these numbers lies between 1 and 2; hence the whole number 1 in each index.

These powers of ten are known as **common logarithms**.

The principle underlying the use of logarithms may be illustrated by the two following simple examples.

$$(i) \ 3 \times 5 = 10^{0.4771} \times 10^{0.6990} = 10^{0.4771+0.6990}, \text{ by (i),} \\ = 10^{1.1761} = 15.$$

$$(ii) \ 3^2 = (10^{0.4771})^2 = 10^{0.9542}, \text{ by (iii),} \\ = 9.$$

EX. 1. *Using the powers of ten given above, find the logarithms of*

(i) 23.1 and (ii) 3.51.

$$(i) \ 23.1 = 231 \div 10 = 3 \times 7 \times 11 \div 10 \\ = 10^{0.4771} \times 10^{0.8451} \times 10^{1.0414} \div 10^1 \\ = 10^{0.4771+0.8451+1.0414-1} = 10^{1.3636}.$$

\therefore To the base 10, the logarithm of 23.1 = 1.3636.

This fact is generally stated in the abbreviated form :

$$\log_{10} 23.1 = 1.3636.$$

$$(ii) \ 351 = 3 \times 3 \times 3 \times 13 = 3^3 \times 13.$$

But $3 = 10^{0.4771}$, so that, the logarithm of 3 to the base 10 is 0.4771,

or, briefly, $\log_{10} 3 = 0.4771$;

$$\therefore \log_{10} 3^3 = 3 \log_{10} 3 = 1.4313$$

Similarly $\log_{10} 13 = 1.1139$

$$\therefore 3 \log_{10} 3 + \log_{10} 13, \text{ or } \log_{10} 351 = 2.5452$$

$$\text{Finally, } 3.51 = 351 \div 100 = 351 \div 10^2 \\ = 10^{2.5452} \times 10^{-2} = 10^{0.5452},$$

$$\text{i.e. } \log_{10} 3.51 = 0.5452.$$

In the statement of a logarithm, it is customary to omit the base unless it is different from 10; thus, the results (i) and (ii) above would generally be written : $\log 23.1 = 1.3636$; $\log 3.51 = 0.5452$.

Logarithms in general use are calculated to four places of decimals, as illustrated above, but for some calculations seven and even ten places may be needed. In certain types of commercial problems it is necessary to use seven places, as will be shewn later.

13-4. The Two Parts of a Logarithm.

It should be clear from the examples already given that the logarithm of every number consists of *two* parts : a whole number and a decimal. In a table of logarithms, only the decimal part is shewn. The following is an extract of a four-figure table.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7

To use the table, note that the first two digits of a given number of four figures, whose logarithm is required, are found in the first left-hand column. The third digit is given in the next set of columns numbered at the top 0, 1, 2, ... 9, and the fourth digit is given in the narrow columns on the right-hand side, headed 1, 2, 3, ... 9.

Suppose the logarithm of 5748 were required. Look along the horizontal row beginning with 57 until the column headed 4 is reached. The number shewn here in the table is 7589. Proceed to the right along the same row until the narrow column headed 8 is found. The number shewn here is 6 ; this must be added to 7589, thus $7589 + 6 = 7595$. Remembering that only the decimal part of

the logarithm is shewn in the table, $\log 5748 =$ a whole number $+0\cdot7595$.

Now 5748 lies between $1000=10^3$ and $10,000=10^4$, so that $\log 5748$ is greater than 3 and less than 4, i.e.

$$\log 5748 = 3\cdot7595.$$

Similarly,

$$\log 574\cdot8 = 2\cdot7595,$$

$$\log 57\cdot48 = 1\cdot7595,$$

and

$$\log 5\cdot748 = 0\cdot7595.$$

Hence, the number of digits to the left of the decimal point in any number determines the whole number part of its logarithm. For this reason the whole number part is called the **characteristic**, whilst the decimal part of a logarithm is known as the **mantissa**, a Latin word meaning a *makeweight*.

It will also be seen that the characteristic of every number greater than unity is just **one less** than the number of digits to the left of the decimal point in that number.

The following simple example will shew how the tables may be used to shorten the work of calculation.

Ex. 2. Calculate, by logarithms, the value of

$$\frac{5\cdot478 \times 586\cdot3}{56\cdot97}.$$

The numbers have been chosen so that the extract from four-figure tables, shewn opposite, may be used. A complete table of four-figure logarithms is given on pages 338-9.

Reading from the table, as explained above :

$$\left. \begin{array}{l} \log 5\cdot478 = 0\cdot7386 \\ \log 586\cdot3 = 2\cdot7681 \end{array} \right\} \begin{array}{l} \text{adding as the given numbers are to be} \\ \text{multiplied.} \end{array}$$

$$\underline{3\cdot5067}$$

$$\log 56\cdot97 = 1\cdot7556$$

$$\text{Subtract} \quad \underline{1\cdot7511} = \log \text{ of the answer.}$$

Looking for the decimal part 0·7511, which will be represented

in the table by 7511, among the logarithms, the row beginning with 56 is reached. Along this row, the nearest number to 7511 is 7505 in the column headed 3; this represents the decimal part of $\log 5630$.

Now $7511 - 7505 = 6$, and along the same row, 6 is found in the narrow columns under 8; hence, 0.7511 is the decimal part of $\log 5638$.

But the characteristic 1 indicates that the number is greater than 10^1 or 10 and less than 10^2 or 100;

\therefore the required number is 56.38.

13-5. The Logarithm of a Number less than Unity.

Suppose the logarithm of 0.5187 were required. From the table, the decimal part of the logarithm of 5187 is 0.7149.

Now $0.5187 = 5.187 \div 10$, so that

$$\log 0.5187 = \log 5.187 - \log 10 = 0.7149 - 1.$$

The subtraction here indicated is not carried out, but the logarithm is written in the form $\bar{1}.7149$, thus denoting briefly $-1 + 0.7149$. $\bar{1}$ is read *bar one*. This device preserves the method of writing down the characteristic for numbers less than unity. For instance

$$\log 518.7 = 2.7149,$$

$$\log 51.87 = 1.7149,$$

$$\log 5.187 = 0.7149,$$

$$\log 0.5187 = \bar{1}.7149.$$

Similarly,

$$\log 0.05187 = \bar{2}.7149,$$

and so on.

Hence, the logarithm of a number less than unity has a negative characteristic.

The advantage of keeping the decimal part always positive renders the same set of tables applicable to all numbers whether greater or less than unity. Note that for 0.5187, the first signi-

ficant figure occupies the *first* place after the decimal and the characteristic of its logarithm is -1 or $\bar{1}$; for 0.05187, the first significant figure occupies the *second* place and the characteristic of its logarithm is $\bar{2}$, and so on. Hence, the following simple rule:

The characteristic of the logarithm of a number less than unity is always negative, and when the number is expressed as a decimal, the characteristic is equal to the number of the place occupied by the first significant figure to the right of the decimal.

Ex. 3. *By the use of logarithms, express a metric tonne in terms of a British ton, given that a metric tonne = 1000 kilograms and one gram = 15.43 grains.*

Before applying logarithms, first obtain the required relation as an ordinary fraction. Thus,

$$\begin{aligned} 1 \text{ metric tonne} &= 1000 \text{ kgm.} = 1000 \times 1000 \text{ gm.} \\ &= 10^6 \text{ gm.} = 10^6 \times 15.43 \text{ grains.} \end{aligned}$$

$$\begin{aligned} \text{Now, from Section 3.4, } 1 \text{ lb. avoirdupois} &= 7000 \text{ grains} \\ &= 7 \times 10^3 \text{ grains.} \end{aligned}$$

$$\begin{aligned} \therefore 1 \text{ metric tonne} &= \frac{10^6 \times 15.43}{7 \times 10^3} \text{ lb.} = \frac{10^3 \times 15.43}{7} \text{ lb.} \\ &= \frac{10^3 \times 15.43}{7 \times 2240} \text{ ton} = \frac{1543}{7 \times 224} \text{ ton.} \end{aligned}$$

Now apply logarithms, using the tables given on pages 338-9:

$$\begin{array}{rcl} \log 1543 & & = 3.1884 = 2 + 1.1884 \\ \log 7 & = 0.8451 & \\ \log 224 & = 2.3502 & \\ \hline & 3.1953 & \end{array} \quad \begin{array}{l} \nearrow 3.1953 = 3 + 0.1953 \\ -1 + 0.9931 = \bar{1}.9931 \end{array}$$

Note that 3.1953 is greater than 3.1884 and, in order to keep the decimal part positive, the characteristic 3 of the upper logarithm is written in the form $2 + 1$. The subtraction then renders the decimal part of the difference positive, whilst $2 - 3$ gives the negative characteristic -1 or $\bar{1}$.

Now from the tables, 0·9931 is the decimal part of the logarithm of 9842. It should be observed that, in looking for this number, 9930 is first found in the horizontal row beginning with 98, and then in the narrow difference columns there are two ones in columns headed 2 and 3 respectively. When this happens, it is usual to take the required difference in the first column reached in traversing the row from left to right. Hence in the present case the number is 9842 and not 9843.

The characteristic $\bar{1}$ indicates a number less than unity with the first significant figure in the first place after the decimal point ;

$$\therefore \bar{1} \cdot 9931 = \log 0 \cdot 9842,$$

so that 1 metric tonne = 0·9842 of a British ton.

13·6. Powers and Roots by Logarithms.

From the index laws established in Sections 13·1 and 13·2, it is comparatively simple to apply logarithms to determine the powers and roots of numbers. In general, representing numbers by letters, when $P = a^n$;

$$\log P = n \times \log a. \dots\dots\dots(\text{viii})$$

When $R = n$ th root of $a = \sqrt[n]{a} = a^{\frac{1}{n}}$, by (vi),

$$\log R = \frac{1}{n} \log a. \dots\dots\dots(\text{ix})$$

The following examples will shew how the method may be practically applied.

Ex. 4. Find, by logarithms, the values of

$$(i) (2 \cdot 947)^3 ; (ii) \sqrt[5]{74 \cdot 73} ; (iii) (0 \cdot 86)^4 ; (iv) \sqrt[7]{0 \cdot 5752}.$$

Taking logarithms in each case :

$$(i) \log (2 \cdot 947)^3 = 3 \times \log 2 \cdot 947 = 3 \times 0 \cdot 4693 = 1 \cdot 4079 = \log 25 \cdot 58.$$

$$\therefore (2 \cdot 947)^3 = 25 \cdot 58.$$

$$(ii) \log \sqrt[5]{74 \cdot 73} = \frac{1}{5} \log 74 \cdot 73 = \frac{1 \cdot 8735}{5} = 0 \cdot 3747 = \log 2 \cdot 370 ;$$

$$\therefore \sqrt[5]{74 \cdot 73} = 2 \cdot 37.$$

$$\begin{aligned}
 \text{(iii) } \log (0.86)^4 &= 4 \times \log 0.86 = 4 \times \bar{1}.9345 \\
 &= 4 \times (-1) + 4 \times 0.9345 = -4 + 3.7380 \\
 &= \bar{1}.7380 = \log 0.547 ; \\
 \therefore (0.86)^4 &= 0.547.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv) } \log \sqrt[7]{0.5752} &= \frac{1}{7} \log 0.5752 = \frac{\bar{1}.7599}{7} = \frac{\bar{7} + 6.7599}{7} \\
 &= -1 + 0.9657 = \bar{1}.9657 = \log 0.924 ; \\
 \therefore \sqrt[7]{0.5752} &= 0.924.
 \end{aligned}$$

Note carefully that, in multiplying a logarithm with a negative characteristic by an integer, the positive and negative parts must be multiplied separately ; thus

$$\bar{1}.9263 \times 13 = (-1) \times 13 + 0.9263 \times 13 = -13 + 12.0419 = \bar{1}.0419.$$

For division by an integer, the negative characteristic must be written in the form *-(dividing integer, or the nearest multiple of it) + the positive number necessary.*

$$\text{Thus } \frac{\bar{2}.8417}{11} = \frac{-11 + 9.8417}{11} = -1 + 0.8947 = \bar{1}.8947.$$

This is much quicker than making the whole logarithm negative, for if this be done, then after division, the resulting negative logarithm has finally to be converted into its equivalent having a positive mantissa and a negative characteristic ;

$$\text{thus } \bar{2}.8417 = -2 + 0.8417 = -1.1583,$$

$$\text{and } -1.1583 \div 11 = -0.1053 = -1 + 1 - 0.1053 = \bar{1}.8947.$$

$$\text{Ex. 5. Evaluate } \frac{\sqrt[5]{67.92} \times (0.8673)^4}{(1.613)^3 \times \sqrt[7]{0.4284}}.$$

From the tables of four-figure logarithms,

$$\log \sqrt[5]{67.92} = \frac{1}{5} \log 67.92 = \frac{1.8320}{5} = 0.3664$$

$$\begin{aligned}
 \log (0.8673)^4 &= 4 \times \log 0.8673 = 4 \times \bar{1}.9382 = \bar{1}.7528 \\
 &\quad \underline{\underline{0.1192}}
 \end{aligned}$$

$$\log (1.613)^3 = 3 \times \log 1.613 = 3 \times 0.2076 = 0.6228$$

$$\log \sqrt[7]{0.4284} = \frac{1}{7} \log 0.4284$$

$$= \frac{\bar{1}.6318}{7} = \frac{\bar{7} + 6.6318}{7} = -1 + 0.9474 = \bar{1}.9474$$

$$\underline{0.5702}$$

The subtraction of 0.5702 from 0.1192 may be carried out as follows :

$$0.1192 = -1 + 1.1192$$

$$0.5702 = \quad 0.5702$$

$$\hline -1 + 0.5490 = \bar{1}.5490 = \log 0.3540.$$

\therefore Required value = **0.354**.

Ex. 6. *A solid cube to weigh approximately 2.76 tons has to be cut from stone weighing 157.2 lb. per cubic foot. Calculate, in feet, the length of its edge.*

Let the length of each edge = x feet,

then volume of cube = x^3 cubic feet,

and its weight = $x^3 \times 157.2$ lb.

But this weight has to be 2.76 tons = 2.76×2240 lb. ;

$$\therefore x^3 \times 157.2 = 2.76 \times 2240,$$

from which

$$x = \sqrt[3]{\frac{2.76 \times 2240}{157.2}}.$$

Taking logarithms :

$$\log 2.76 = 0.4409$$

$$\log 2240 = 3.3502$$

$$\hline 3.7911$$

$$\log 157.2 = 2.1965$$

$$\hline 1.5946 \text{ Dividing by 3 to find the cube root.}$$

$$\hline 0.5315$$

$$= \log 3.400.$$

\therefore Length of edge = **3.4 feet**.

EXERCISES 13

No logarithms are to be used for Exercises 1-6.

1. Express as powers of 2 and simplify : $(2^7)^2 \times 16^{-5} \times 256^{\frac{3}{4}}$.

2. Simplify $10^{0.6} \times 10^{1.62} \div 10^{-0.78}$.

3. Find the simplest value of

$$\frac{10^{1.61} \times 10^3 \times 10^{2.34}}{10^{4.67} \times 10^{3.28}}.$$

4. By applying the laws of indices, simplify the expression

$$\frac{8^6 \times 6^8}{12^{12}},$$

giving the result as a vulgar fraction in its lowest terms. (U.L.C.I.)

5. Simplify the following :

$$(a) \frac{x^3 \times x^6}{x^4}; \quad (b) x^0; \quad (c) 3^{-3}; \quad (d) 16^{\frac{1}{2}}. \quad (\text{U.L.C.I.})$$

$$6. \text{ Simplify } (a) \sqrt{\frac{x^7}{x^8 \times x^{-7}}}; \quad (b) 8^{-\frac{2}{3}}; \quad (c) \log \frac{1}{1000}. \quad (\text{U.L.C.I.})$$

In the following exercises, use four-figure logarithm tables. Where necessary, take $\log \pi = 0.4972$.

7. Find the value of $\frac{41.86 \times 81.34 \times 3.643}{3.49 \times 48.97}$.

8. Evaluate $\frac{457.2 \times 10.03 \times 9.637}{8.632 \times 15.58 \times 32.67}$.

9. Calculate the value of $\frac{3.14 \times 68.46 \times 468.2}{32.34 \times 92.86}$.

10. The following calculation has to be made to determine the amount of an insurance bonus :

$$\text{£} \frac{83.6 \times 6.27 \times 19.32}{1.651 \times 326.2}.$$

Find the bonus.

11. Calculate the number of chains equivalent to one kilometre, using only the relation 1 inch = 2.54 centimetres.

12. Calculate the value of $(0.119 \times 3.12) \div (1.31 \times 0.0079)$.

13. Simplify, by use of logarithms :

$$(i) 23.66 \times 20.36 \times 0.3681 ; \quad (ii) (1.745)^4 ;$$

$$(iii) \frac{8.617 \times 0.0922}{0.5468} . \quad (U.L.C.I.)$$

14. Evaluate $\frac{63.9 \times 47.4 \times 75.8}{7.7 \times 84.3 \times 869} .$

15. Calculate the value of $(3.612)^3 \div (86.17 \times 6.327) .$

16. Find the value of $\frac{(4.867)^2 \times 0.3782}{14.76 \times \sqrt[3]{0.7358}} . \quad (C.P.)$

17. Simplify, by use of logarithms :

$$(a) \frac{72.51 \times 226.1}{904.9} ; \quad (b) (0.3971)^2 \times \sqrt[3]{6.904} . \quad (U.L.C.I.)$$

18. A cylindrical thread of mercury is 5.72 cm. long and weighs 4.13 grams. What is the diameter of the thread if one cubic centimetre of mercury weighs 13.6 grams?

19. Calculate the value of $\frac{\pi \times (784.1)^{\frac{1}{4}} \times (15.37)^2}{10.52 \times \sqrt[3]{3378}} .$

20. Calculate $\sqrt[3]{3.416} \div \sqrt[5]{6.327} . \quad (R.S.A.)$

21. Find the value of $\sqrt[7]{12.43} \div \sqrt[5]{8.264} . \quad (R.S.A.)$

22. Calculate the value of $\sqrt[7]{161.4} \div \sqrt[5]{57.24} . \quad (R.S.A.)$

23. (i) Calculate $\sqrt[3]{17.24} \div \sqrt[5]{26.35} .$

(ii) Find, to four significant figures, the weight in tons of $2\frac{1}{2}$ inches of snow on a square mile of country. Twelve inches of snow melt to one inch of water and one cubic foot of water weighs 62.3 lb. (R.S.A.)

24. During a storm 0.34 inch of rain fell. Calculate how many tons per acre this was, given that one cubic foot of rain water weighs 62.3 lb.

25. Given that 1 hectare = 10,000 square metres and that 1 metre = 39.37 inches, calculate by logarithms (i) the number of acres in one hectare, and (ii) the number of square kilometres equivalent to one square mile. (U.L.C.I.)

26. Calculate the value of $\sqrt[5]{2.415} \times \sqrt[7]{0.296} \div (0.824)^4 . \quad (R.S.A.)$

27. By means of logarithms, (a) find the value of

$$\sqrt{16.81 \times 0.6783 \div (1.163)^2},$$

and (b) express one cubic metre in cubic yards, given that 1 metre = 39.37 inches. (U.L.C.I.)

28. Calculate the value of $(1.146)^{2.3} \div (0.876)^5$. (R.S.A.)

29. From the formula $A = P \left(1 + \frac{r}{100} \right)^n$, calculate the value of r when $A = 1148$, $P = 436.5$ and $n = 28$.

30. A powder is sold in rectangular packets measuring 3.4 in. by 2.8 in. by 1.3 in. For export trade, the powder must be packed in sealed cylindrical tins 5.7 in. high, each tin to hold the powder contained in three packets. Calculate, to the nearest inch, the diameter of a tin.

CHAPTER XIV

THE APPLICATION OF LOGARITHMS TO COMMERCIAL CALCULATIONS

14.1. The Calculation of Simple Interest.

WHERE the time period involves a number of days, logarithms may conveniently be applied to the calculation of simple interest, as in the following example.

Ex. 1. Calculate, to the nearest penny, the simple interest on £277 for 2 years 131 days at $3\frac{1}{2}$ per cent. per annum.

$$2 \text{ years } 131 \text{ days} = (730 + 131) \text{ days} = 861 \text{ days} = \frac{861}{365} \text{ years.}$$

∴ By the method explained in Section 6.2,

$$\text{S.I.} = \pounds \frac{277 \times 7 \times 861}{2 \times 100 \times 365} = \pounds \frac{1669.479}{73} = \pounds 22 \text{ } 17\text{s. } 5\text{d.},$$

to the nearest penny.

The long multiplication and division may be dispensed with by the use of logarithms, thus :

log 277 = 2.4425	log 200 = 2.3010
log 7 = 0.8451	log 365 = 2.5623
log 861 = 2.9350	<u>4.8633</u>
6.2226	
4.8633	
<u>1.3593</u>	

1.3593 = log 22.87

∴ S.I. = £22.87 = £22 17s. 5d., to the nearest penny.

In using four-figure tables, the result expressed in pounds will only contain four figures, so that, unless the interest is less than £10, the fraction of £1 will not be given to the three places essential for finding an answer correct to the nearest penny. In most cases, therefore, a discrepancy is likely to occur between the true value

of the interest and that calculated by four-figure logarithms, particularly when the sums involved are large. This difficulty is removed by using logarithms calculated to seven places of decimals. The use of such logarithms is even more important in connection with the case of compound interest.

14·2. The Calculation of Compound Interest.

The method of calculating compound interest year by year has been explained in Chapter VIII, but the process may be materially shortened by the use of logarithms, and especially so when the period is long. Consider very carefully the following example.

Ex. 2. *To what sum will £756 amount in eight years if invested at $3\frac{1}{4}$ per cent. per annum, compound interest, the interest being added yearly? Give the amount to the nearest penny.*

By the method of Section 8·1, the working would appear thus :

£		£
756·00000	→ $A_4 =$	859·17575
22·68		25·77525
1·89		2·14793
$A_1 = 780·57$		$A_5 = 887·09893$
23·4171		26·61294
1·95142		2·21774
$A_2 = 805·93852$		$A_6 = 915·92961$
24·17814		27·47787
2·01484		2·28982
$A_3 = 832·13150$		$A_7 = 945·69730$
24·96393		28·37091
2·08032		2·36424
$A_4 = 859·17575$		$A_8 = 976·43245 = \text{£}976 \text{ 8s. 8d.}$

Note the length of the calculation by the ordinary method.

From Section 8·4, if £ P amounts to £ A in n years at $r\%$ per annum,

$$A = PR^n, \text{ where } R = 1 + \frac{r}{100}.$$

In the present case, $R = \frac{413}{400}$, so that $A = 756 \times \left(\frac{413}{400}\right)^8$.

From four-figure logarithms :

$$\log 413 = 2.6160$$

$$\log 400 = 2.6021$$

$$0.0139$$

$$8 \times \log \quad = 0.1112$$

$$\log 756 = 2.8785$$

$$\therefore \log A = 2.9897 = \log 976.6$$

so that the required amount is

$$£976.6 = £976 \text{ 12s.},$$

which is too large by 3s. 4d.

Hence, this result is not correct to the nearest penny because the fourth figure of each logarithm is unreliable. The error is made even greater by the fact that $\log R$ has been multiplied by 8. Further, as in the case of simple interest, unless the result to be found is less than £10, the number of decimal places required to give an answer to the nearest penny will not be determined. Errors would be considerably increased with larger sums and for longer periods. It should therefore be clear that, whilst the application of logarithms provides a short method of calculation, however many years there may be in the period, more exact logarithms are needed.

14.3. Seven-figure Logarithms.

As a general rule, logarithms calculated to seven places of decimals will be sufficiently accurate to give most results to the nearest penny. In such logarithms, the differences are considerably smaller and vary so much that they cannot be usefully tabulated as in four-figure tables. They are therefore usually

given for numbers from 1000 to 9999, of which the following is a sample :

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
136	1335389	1338581	1341771	1344959	1348144	1351327	1354507	1357685	1360861	1364034
137	67206	70375	73541	76705	79867	83027	86184	89339	92492	95643
138	98791	1401937	1405080	1408222	1411361	1414498	1417632	1420765	1423895	1427022
139	1430148	33271	36392	39511	42628	45742	48854	51964	55072	58177
140	61280	64381	67480	70577	73671	76763	79853	82941	86027	89110
141	92191	95270	98347	1501422	1504494	1507564	1510633	1513699	1516762	1519824
142	1522883	1525941	1528996	32049	35100	38149	41195	44240	47282	50322
143	53360	56396	59430	62462	65492	68519	71544	74568	77589	80608
144	83625	86640	89653	92663	95672	98678	1601683	1604685	1607686	1610684

To illustrate the effectiveness of these seven-figure tables, Ex. 2 will be re-calculated.

Ex. 3. *By the use of the seven-figure logarithms given, calculate the amount of £756 for eight years at $3\frac{1}{4}$ per cent. per annum compound interest.*

From the formula : $A = PR^n$, $R = 1 + 0.0325 = 1.0325$
so that $A = 756 \times (1.0325)^8$.

Now, since the tables usually give the logarithms of numbers up to four figures, 1.0325 may be expressed either as 5×0.2065 or $413/400$; hence from the seven-figure tables shown on pages 340-57,

$$\log 1.0325 = \log 5 + \log 0.2065 = 0.6989700 + \bar{1}.3149201 = 0.0138901$$

$$\therefore \log 1.0325 = 0.0138901$$

$$8 \times \log 1.0325 = 0.1111208$$

$$\log 756 = 2.8785218$$

$$\therefore \log A = 2.9896426$$

Reference again to the tables shows that A lies between 976.4 and 976.5, i.e.

$$\log A = \log (976 + \epsilon),$$

where ϵ is a number less than unity which has to be determined.

To do this, the method of proportional differences must be applied, thus :

$\begin{array}{r} \log (976 + \epsilon) = 2.9896426 \\ \log 976 \quad \quad = 2.9894498 \\ \hline \text{Difference for } \epsilon = 0.0001928 \end{array}$	$\begin{array}{r} \log 977 = 2.9898946 \\ \log 976 = 2.9894498 \\ \hline \text{Difference for } 1 = 0.0004448 \end{array}$
--	--

Hence $\epsilon : 1 = 0.0001928 : 0.0004448 = 1928 : 4448$, from which $\epsilon = 0.4335$.

$\therefore A = £976.4335 = £976 \text{ 8s. 8d.}$, which is correct to the nearest penny.

14.4. Calculation of Depreciation.

In a similar way, logarithms may be applied to calculations of depreciation.

Ex. 4. *Some machinery costing £9562 is estimated, at the end of each year, to depreciate by 8.5 per cent. of its value at the beginning of the year. Calculate, to the nearest penny, its value at the end of nine years.*

From Section 8.7, if $£P$ = initial value, $£V_9$ = value at end of 9 years and r = rate per cent. per annum of depreciation, then

$$V_9 = PD^9, \text{ where } D = 1 - \frac{r}{100}.$$

Hence, in this case, $D = 1 - 0.085 = 0.915$;

$$\therefore V_9 = 9562 \times (0.915)^9.$$

From the table of seven-figure logarithms :

$$\log 0.915 = \bar{1}.9614211$$

$$\therefore 9 \times \log 0.915 = \bar{1}.6527899$$

and

$$\begin{array}{r} \log 9562 = 3.9805487 \\ \hline \therefore \log V_9 = 3.6333386 = \log (4298 + \epsilon), \end{array}$$

where ϵ is a fraction to be determined by the method of proportional differences.

Now	$\log (4298 + \epsilon) = 3.6333386$	$\log 4299 = 3.6333674$
	$\log 4298 = 3.6332664$	$\log 4298 = 3.6332664$
	Difference for $\epsilon = 0.0000722$	Difference for 1 = 0.0001010

$$\therefore \epsilon : 1 = 0.0000722 : 0.0001010 = 722 : 1010,$$

from which $\epsilon = 0.715$,

$$\therefore V_9 = £4298.715$$

$= £4298 \text{ 14s. 4d., to the nearest penny.}$

14.5. Calculation of Rate.

Logarithms may conveniently be applied to calculate the rate per cent. per annum, especially when the time period is long.

Ex. 5. *In the third issue of Savings Certificates, the initial cost of a certificate was 16s. and in ten years its value became 24s. Calculate, to two significant figures, the average rate of compound interest per cent. per annum.*

Here $A = 24\text{s.}$, $P = 16\text{s.}$ and $n = 10$.

Hence, since $A = PR^n$, where $R = 1 + \frac{r}{100}$,

$$24 = 16 \times R^{10},$$

or
$$R^{10} = \frac{24}{16} = \frac{3}{2} = 1.5.$$

$$\therefore 10 \times \log R = \log 1.5 = 0.1761 \text{ or } 0.1760913,$$

so that
$$\log R = 0.01761 \text{ or } 0.0176091$$

$$= \log 1.041 \text{ or } \log 1.04138$$

Hence
$$r = 4.1 \text{ or } 4.138,$$

i.e., to two significant figures,

the rate per cent. per annum = 4.1.

Note that, as the result is only required to two significant figures, four-figure tables will, in general, suffice when the rate is less than 10. For a greater degree of accuracy, and, generally, when the rate exceeds 10, seven-figure tables must be used.

14.6. Calculation of Time.

In this case, the calculation differs fundamentally from those already dealt with in so far that the answer is actually an index and therefore a *logarithm*.

For since $A = PR^n$,

$$\therefore \log A = \log P + n \log R,$$

from which
$$n = \frac{\log A - \log P}{\log R}.$$

Hence, the value of n is the quotient of two logarithms.

Now, from the rule for the approximate division of decimals deduced from Ex. 10, p. 32, when an answer is required to m significant figures, the division can be carried out with $(m+1)$ significant figures. As m will seldom exceed 3, a minimum of four figures will generally be sufficient with which to carry out the division, so that *four-figure tables may be used when the required answer is to be found to the nearest year or nearest tenth of a year, provided the period does not exceed 100 years*. Where, however, the time has to be given to the nearest day, seven-figure logarithms must be applied.

Ex. 6. £538 is deposited in a bank to accumulate at $2\frac{3}{4}$ per cent. per annum compound interest. Find in how many years, to the nearest year, it will become £1000.

Here
$$R = 1 + \frac{2\frac{3}{4}}{100} = \frac{411}{400}.$$

$$\therefore 1000 = 538 \times \left(\frac{411}{400}\right)^n,$$

or $\left(\frac{411}{400}\right)^n = \frac{1000}{538}$, where n is the required number of years.

Now n will not exceed 100, so that it will only contain two figures at most; hence, a minimum of three figures must be used in the division, i.e. four-figure logarithms are sufficient. It is for

this reason that the value of R is expressed as an ordinary fraction for, as a decimal, it requires five places.

$$\begin{array}{rcl} \log 411 & = & 2.6138 \\ \log 400 & = & 2.6021 \\ \hline & & 0.0117 \end{array} \qquad \begin{array}{rcl} \log 1000 & = & 3.0000 \\ \log 538 & = & 2.7308 \\ \hline & & 0.2692 \end{array}$$

$$\therefore n \times 0.0117 = 0.2692,$$

or
$$n = \frac{0.2692}{0.0117} = \frac{2692}{117} = 23.00\dots$$

\therefore Required time, to the nearest year = 23 years.

Ex. 7. On June 1st, 1920, a sum of money was invested at $2\frac{1}{2}$ per cent. per annum compound interest. Find the approximate date upon which the sum will become just doubled.

If $\pounds P$ = the sum, n = number of years in which the amount becomes $\pounds 2P$, then

$$2P = PR^n = P(1.025)^n,$$

so that
$$(1.025)^n = 2,$$

or
$$n \times \log 1.025 = \log 2;$$

$$\therefore n \times 0.0107239 = 0.3010300,$$

using seven-figure logarithms as the result is required to the nearest day.

Hence,
$$n = \frac{3010300}{107239} = 28.07094\dots$$

$$= 28 \text{ years } 25.89 \text{ days}$$

$$= 28 \text{ years } 26 \text{ days, to the nearest day.}$$

Hence the sum will have doubled itself on **June 27th, 1948.**

It should be pointed out here that n has been calculated on the assumption that the formula, $A = PR^n$, remains valid for fractional values of n . To test the result found, note that the time $\pounds 1$ becomes $\pounds 2$ lies between 28 years and 29 years.

Now the amount of $\pounds 1$ in 28 years at $2\frac{1}{2}\%$ per annum = $\pounds (1.025)^{28}$ = $\pounds 1.9965$, which is $\pounds 0.0035$ less than $\pounds 2$. Hence, since the interest for a fraction of a year is that fraction of the interest for a whole

year, the number of days in which £1·9965 will gain £0·0035 as interest is $\frac{0·0035 \times 100 \times 365}{1·9965 \times 2·5} = 25·59$.

Thus the discrepancy is 0·3 of a day, but fractions of a day are not considered in practice, so that the error may be neglected.

14·7. Time to reach a given Value by Depreciation.

This type of calculation is precisely similar to that of compound interest, but here negative characteristics are involved.

Ex. 8. *An asset originally worth £6791 depreciates each year by 8·5 per cent. of its value at the beginning of the year. In how many years will its value be reduced to £1500?*

From Section 8·7, $V_n = PD^n$, and $V_n = £1500$, $P = £6791$,

$$D = 1 - \frac{8·5}{100} = 0·915.$$

$$\therefore 1500 = 6791 \times (0·915)^n,$$

or

$$(0·915)^n = 1500 \div 6791.$$

Taking seven-figure logarithms,

$$n \times \log 0·915 = \log 1500 - \log 6791,$$

or

$$n \times (\bar{1}·9614211) = 3·1760913 - 3·8319337 = \bar{1}·3441576 ;$$

$$\therefore n = \frac{\bar{1}·3441576}{\bar{1}·9614211} = \frac{-1 + 0·3441576}{-1 + 0·9614211} = \frac{-0·6558424}{-0·0385789} = \frac{6558424}{385789} \\ = 17·00... .$$

\therefore Required time = **17 years.**

Note that, to perform the division, logarithms with negative characteristics must be made wholly negative.

EXERCISES 14

For the calculations involved in the following exercises, logarithms must be used. Answers in money should be given to the nearest penny unless stated otherwise.

1. What will £863 amount to in seven years at $2\frac{1}{2}$ per cent. per annum compound interest?

2. Calculate the compound interest on £486 for eight years at $3\frac{1}{4}$ per cent. per annum.

3. An asset originally worth £8520 depreciates each year by 18 per cent. of its value at the beginning of the year. Calculate its value at the end of nine years.

4. Calculate, to the nearest shilling, the compound interest on £1000 for 12 years at 3 per cent. per annum reckoned half-yearly.
(R.S.A.)

5. £3295 is invested at $4\frac{1}{2}$ per cent. per annum compound interest. By use of tables, find (a) the increase for 8 years, (b) the increase for 9 years and (c) the increase during the 9th year.
(U.L.C.I.)

6. A sum of £863 amounts to £1000 in five years at compound interest; find the rate per cent. per annum.

7. In how many years will £943 amount to £1199 at $3\frac{1}{2}$ per cent. per annum compound interest?

8. Calculate the value of property at the end of nine years if its original value was £14,370, depreciation being reckoned at 7 per cent. per annum.

9. The population, in thousands, of a certain town was 9201 in 1926 and 9612 in 1936. Calculate the average percentage increase per annum.

(Note that the increase of a population follows the compound interest law.)

10. To allow for depreciation of machinery costing £1000, 15 per cent. of its value is deducted at the end of the first year. At the end of the second year $12\frac{1}{2}$ per cent. of its value at the beginning of the year is deducted and, at the end of each succeeding year, 8 per cent. of its value at the beginning of the year is deducted. After how many years will its value be first entered as less than £200 and what will then be its value?
(L.Ch.C.)

11. £576 is invested for 18 years at $4\frac{1}{2}$ per cent. per annum compound interest. Find, by means of tables, (a) the total increase, (b) the increase during the last 3 years. (U.L.C.I.)

12. A sum of money is invested at 3 per cent. per annum compound interest reckoned half-yearly. After what interval of time will the interest for the first time exceed 50 per cent. of the original sum invested? (R.S.A.)

13. A sum of £3752 is left to accumulate at compound interest for 19 years. The total interest added for that period is £1974. Calculate the rate per cent. per annum at which the interest is added.

14. A piece of machinery purchased in the year 1900 and entered in the books of a certain manufacturing company for that year at its purchase price, had a book value of £231 in the year 1913 and of £71 in 1936. Every year since 1900 a certain fixed percentage of the book value entered for the preceding year has been written off for depreciation. Calculate the original purchase price. (R.S.A.)

15. Some property originally valued at £7857 is allowed to deteriorate and, 13 years later, its value is estimated at £697. Calculate the average percentage depreciation per annum to the nearest whole number.

16. Find the amount of £762 for eight years at $3\frac{1}{2}$ per cent. per annum compound interest.

17. £540 at compound interest for seven years amounts to £686 17s. 7d. Find the rate per cent. per annum. (B.M.I.)

18. In 1921 the population, in thousands, of Greater London was 7480, whilst in 1931 it had become 8204. Calculate the average percentage increase per year, correct to two significant figures.

19. At what rate per cent. per annum, compound interest, will £340 amount to £528 in ten years? (R.S.A.)

20. Calculate the number of years in which £500 will amount to £834 4s. at $2\frac{3}{4}$ per cent. per annum compound interest.

21. £100 was deposited at the end of 1934 in the Post Office Savings Bank, which pays interest at $2\frac{1}{2}$ per cent. per annum, the interest being added to the principal on December 31st of each year. If the money is left to accumulate, in what year will the interest added on December 31st first exceed £4? (R.S.A.)

22. Find the rate per cent. per annum at which £2380 will amount to £5027 at compound interest in 17 years.

23. A machine was bought for £2500. Each year 10 per cent. was written off the preceding year's valuation as depreciation. What was the valuation at the end of nine years? (U.L.C.I.)

24. Calculate in how many years an asset worth £7692 will fall in value to £828, reckoning depreciation at 13 per cent. per annum.

25. A man has £59 12s. in the Post Office Savings Bank. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, how many years will it be before this deposit amounts to £100?

26. The compound interest on £555 for 12 years is £283 14s. Calculate the compound interest on the same sum at the same rate per cent. per annum for 19 years.

CHAPTER XV

ARITHMETICAL SERIES AND ITS APPLICATION TO THE INSTALMENT PLAN

15.1. Arithmetical Progression.

MANY problems are met with in commercial transactions which involve series or sequences of numbers formed according to some fixed condition. One of the simplest of these series is that in which the difference between each pair of consecutive numbers is the same. For instance, in the natural sequence 1, 2, 3, 4, ... each pair of consecutive numbers differs by unity. Similarly, 3, 10, 17, 24, ... and 87, 78, 69, 60, ... are examples of the same type, for pairs of consecutive numbers differ by 7 and -9 respectively.

Each of such series is known as an **Arithmetical Progression**, these words being denoted briefly by their initial letters, **A.P.** The constant difference between each pair of consecutive numbers, or **terms**, as they are usually called, is described as the **Common Difference**.

In the series 3, 10, 17, 24, ... , which is an A.P.,

the 1st term = 3,

the 2nd term = $10 = 3 + 7$,

the 3rd term = $17 = 10 + 7 = 3 + 2 \times 7$,

the 4th term = $24 = 17 + 7 = 3 + 3 \times 7$,

and so on.

In general, if a is the first term and d the common difference, then the

$$\text{nth term} = a + (n - 1)d. \dots\dots\dots(i)$$

Ex. 1. Find

(a) the 14th term of the series 2, 7, 12, ... ;

(b) the 10th term of the series $2\frac{3}{4}$, 4, $5\frac{1}{4}$, ... ;

(c) the last term of the series 85, 78, 71, ... to 13 terms.

In each case, it should be noticed that before (i) can be used, it is necessary to find the common difference, d .

(a) Here $d = 7 - 2 = 5$, which is the same as $12 - 7$.

$$\therefore \text{14th term} = 2 + 13 \times 5 = 2 + 65 = 67.$$

(b) In this case, $d = 4 - 2\frac{3}{4} = 1\frac{1}{4}$. Note that $5\frac{1}{4} - 4 = 1\frac{1}{4}$ also.

$$\therefore \text{10th term} = 2\frac{3}{4} + 9 \times 1\frac{1}{4} = 2\frac{3}{4} + 11\frac{1}{4} = 14.$$

(c) Here it is evident that the last term is the 13th and

$$d = 78 - 85 = -7.$$

Note that, to find d , the *first term must be subtracted from the second*, or the second from the third, etc., for the common difference is the number which must be *added* to one term to form the next. Thus in a series of positive numbers in ascending order of magnitude, the common difference is positive, whilst in the case of numbers in descending magnitude, like those of the present example, the common difference is negative.

$$\begin{aligned} \text{Hence the last term} &= \text{the 13th term} = 85 + 12 \times (-7) \\ &= 85 - 84 = 1. \end{aligned}$$

15.2. The sum of an A.P.

An important problem is to find the sum of a series of numbers in A.P., and this is quite easily done.

Suppose the sum of 12 terms of the series 3, 10, 17, ... be required. Write half the number of terms in order in one line, the remaining half underneath them in the reverse order and add, thus :

$$\begin{array}{rccccccccc} \longrightarrow & & 3 & + & 10 & + & 17 & + & 24 & + & 31 & + & 38 & & \\ & & 80 & + & 73 & + & 66 & + & 59 & + & 52 & + & 45 & & \longleftarrow \\ \hline & & 83 & + & 83 & + & 83 & + & 83 & + & 83 & + & 83 & & \end{array}$$

In this way it will be seen that the sums of the 1st and last terms, the second and the last-but-one terms, ... are all the same. Hence, the sum of 12 terms is $\frac{1}{2} \cdot 12 \times 83 = 6 \times 83 = 498$.

This is true of all series in A.P., and to find the constant sum of each pair of terms, all that is necessary is to add the first to the last terms. This sum multiplied by half the number of terms gives the required sum. To express this rule as a formula, let a , l , denote the first and last terms of an A.P. containing n terms, then their sum S_n is given by

$$S_n = \frac{n}{2} \times (a + l). \dots\dots\dots(ii)$$

Note that if l is not given, the common difference d must be found and l determined from (i) of Section 15·1.

It should be observed that $\frac{1}{2}(a + l)$ is actually the average of the n numbers, for this average

$$= \frac{\text{Sum of numbers}}{n} = \frac{S_n}{n} = \frac{a + l}{2};$$

\therefore the sum of n numbers in A.P. = $\frac{1}{2}n \times$ their average.

Ex. 2. Find the sums of the following series in A.P. :

(a) 5, $8\frac{1}{2}$, 12, ... to 18 terms ;

(b) 7, $11\frac{1}{4}$, $15\frac{1}{2}$, ... 58.

(a) Here the last term, which is the 18th, must be found.

Now $d = 8\frac{1}{2} - 5 = 3\frac{1}{2}$,

$$\therefore \text{18th term} = 5 + 17 \times 3\frac{1}{2} = 5 + 59\frac{1}{2} = 64\frac{1}{2}.$$

Hence, sum of 18 terms = $\frac{1}{2} \cdot 18 \times (5 + 64\frac{1}{2}) = 9 \times 69\frac{1}{2} = 625\frac{1}{2}$.

(b) In this series, the number of terms must be determined.

Let 58 be the n th term ; then since $d = 11\frac{1}{4} - 7 = 4\frac{1}{4}$,

$$58 = 7 + (n - 1) \times 4\frac{1}{4} = 7 + 4\frac{1}{4} \cdot n - 4\frac{1}{4} = 2\frac{3}{4} + 4\frac{1}{4} \cdot n ;$$

$$\therefore 4\frac{1}{4} \cdot n = 58 - 2\frac{3}{4} = 55\frac{1}{4},$$

from which

$$n = 55\frac{1}{4} / 4\frac{1}{4} = 221 / 17 = 13.$$

$$\therefore \text{Sum of 13 terms} = \frac{1}{2} \times 13 \times (7 + 58) = \frac{1}{2}(13 \times 65) = 422\frac{1}{2}.$$

Ex. 3. *A clerk is engaged at a commencing salary of £104 per annum, rising by yearly increments of £15. Find (a) in how many years his salary will become £344 per annum, and (b) the total amount he will have received up to the end of that year.*

(a) It will be obvious that his salary in succeeding years will be the series, £104, £119, £134, ... , which is an A.P. whose common difference is £15.

Hence, £344 is a certain term in the series which has to be found.

Let n = the number of this term, then

$$344 = 104 + (n - 1) \times 15 = 104 + 15n - 15 = 89 + 15n.$$

$$\therefore 15n = 344 - 89 = 255,$$

from which

$$n = 255 \div 15 = 17.$$

\therefore His salary will be £344 per annum in the 17th year.

(b) The total amount of £ S received in 17 years is the sum of the A.P. taken to 17 terms,

$$\text{i.e. } S = \frac{1}{2} \cdot 17 \times (104 + 344) = \frac{1}{2} \cdot 17 \times 438 = 17 \times 219 = 3723.$$

$$\therefore \text{Total sum received in 17 years} = \text{£}3723.$$

15.3. The Instalment Plan of Payment.

In recent times the principle of deferred payments has rapidly developed. Almost any article may now be procured by paying a comparatively small deposit at the time of purchase and then clearing the balance owing by a number of equal payments made at regular intervals. The purchaser thus gets, in effect, a loan which decreases as the instalments are paid off. Interest is therefore charged for this accommodation and, where the period is relatively short, simple interest is reckoned, whilst, for a long period, the interest is calculated on the principle of compound interest, as will be seen later.

Problems of the simple interest type involve an important application of A.P., as the following example will shew.

Ex. 4. *An article priced at £18 may be bought by paying a deposit of £3 at the time of purchase together with nine monthly payments of £1 15s. per month. Calculate the rate per cent. per annum of simple interest charged.*

The purchaser pays altogether $£3 + £(1\frac{3}{4} \times 9) = £(3 + 15\frac{3}{4}) = £18\frac{3}{4}$.

\therefore Extra cost = $£18\frac{3}{4} - £18 = £\frac{3}{4}$ or 15s., which is the interest charged by the vendor.

Now, after paying the deposit of £3, the balance of £15 is still owing, and may be regarded as a loan for one month.

Similarly, after the first monthly payment of £1 15s., the loan becomes £13 5s., and so on.

Hence, the nine monthly loans are

$$£15, £13\frac{1}{4}, £11\frac{1}{2}, £9\frac{3}{4}, £8, £6\frac{1}{4}, £4\frac{1}{2}, £2\frac{3}{4}, £1.$$

These loans are equivalent to a single loan for one month of a sum equal to their total. The numbers form an A.P. whose common difference is $1\frac{3}{4}$.

$$\begin{aligned}\therefore \text{Equivalent loan for one month} &= £(15 + 13\frac{1}{4} + 11\frac{1}{2} + \dots + 1) \\ &= £\{\frac{1}{2} \cdot 9 \times (15 + 1)\} = £72.\end{aligned}$$

Hence the S.I. on this loan for one month at $r\%$ per annum

$$= £ \frac{72 \times r}{12 \times 100} = £ \frac{3 \times r}{50}.$$

But this is 15s. or $£\frac{3}{4}$,

$$\therefore \frac{3 \times r}{50} = \frac{3}{4}, \text{ from which } r = 12\frac{1}{2}.$$

$$\therefore \text{Rate of S.I. charged} = 12\frac{1}{2}\% \text{ per annum.}$$

15.4. The General Case.

In practice, a formula is generally used for the *average* loan over the period; this is sometimes known as the **Equated Amount**. The formula is readily found as follows.

Let $£P$ be the price, $£a$ be the deposit, $£p$ be each monthly instalment and n be the number of payments; then the monthly

loans in £s are

$$P - a, P - a - p, P - a - 2p, \dots P - a - (n - 1)p.$$

$$\begin{aligned} \therefore \text{Equivalent loan} &= \text{£}\{P - a + (P - a) - p + \dots (P - a) - (n - 1)p\}, \\ &= \text{£}\{n(P - a) - (1 + 2 + 3 + \dots + \overline{n - 1})p\} \\ &= \text{£}\{n(P - a) - \tfrac{1}{2}(n - 1)(1 + n - 1)p\} \\ &= \text{£}\{n(P - a) - \tfrac{1}{2}n(n - 1)p\}. \end{aligned}$$

Now since there are n payments, the average loan is

$$\frac{1}{n} \text{£}\{n(P - a) - \tfrac{1}{2}n(n - 1)p\} = \text{£}\{P - a - \tfrac{1}{2}(n - 1)p\}.$$

But total payment made by purchaser = £($a + np$).

\therefore remembering each expression represents £s,

$$\text{S.I.} = a + np - P,$$

from which

$$P = a + np - \text{S.I.}$$

Substituting this value of P in the expression already found for the average loan, this becomes

$$\begin{aligned} &\text{£}\{(a + np - \text{S.I.}) - a - \tfrac{1}{2}(n - 1)p\} \\ &= \text{£}\{\tfrac{1}{2}(n + 1)p - \text{S.I.}\}. \end{aligned}$$

Hence, the equated amount or average loan = £ $\{\tfrac{1}{2}(n + 1)p - \text{S.I.}\}$

Applying this formula to the problem of Ex. 4, the average loan

$$= \text{£}\{\tfrac{1}{2}(9 + 1) \times 1 \tfrac{3}{4} - \tfrac{3}{4}\} = \text{£}(8 \tfrac{3}{4} - \tfrac{3}{4}) = \text{£}8.$$

Hence, £8 is the average loan for 9 months,

\therefore S.I. on £8 for 9 months at $r\%$ per annum

$$= \text{£} \frac{8 \times 9 \times r}{12 \times 100} = \text{£} \frac{3 \times r}{50},$$

and since this is £ $8 \tfrac{3}{4}$, the value of $r = 12 \tfrac{1}{2}$, as before.

Ex. 5. *A man borrows £85 on the condition that he repays it by 21 monthly payments of £4 10s. each, the first to be paid one month from the date of receiving the loan. Calculate the rate per cent. per annum of simple interest charged.*

$$\text{Total sum to be repaid} = £(4\frac{1}{2} \times 21) = £94\frac{1}{2}.$$

$$\therefore \text{S.I. charged} = £94\frac{1}{2} - £85 = £9\frac{1}{2}.$$

Now, from the formula just established,

$$\begin{aligned} \text{average loan in } £ &= \frac{1}{2}(n+1)p - \text{S.I.} = \frac{1}{2}(21+1) \times 4\frac{1}{2} - 9\frac{1}{2} \\ &= 49\frac{1}{2} - 9\frac{1}{2} = 40. \end{aligned}$$

$$\therefore \text{S.I. on } £40 \text{ for 21 months at } r\% \text{ per annum} = £9\frac{1}{2},$$

$$\text{i.e.} \quad \frac{40 \times 21 \times r}{12 \times 100} = \frac{19}{2},$$

$$\text{from which} \quad r = \frac{19 \times 12 \times 100}{2 \times 40 \times 21} = \frac{95}{7} = 13\frac{4}{7}.$$

$$\therefore \text{Required rate} = 13\frac{4}{7}\% \text{ per annum.}$$

EXERCISES 15

1. Sum the series, $1\frac{1}{4}, 1\frac{3}{4}, 2\frac{1}{4}, \dots$ to 17 terms.
2. Find the 22nd term and the sum of 22 terms of the arithmetic progression, $3\frac{1}{3}, 3\frac{3}{4}, 4\frac{1}{6}, \dots$. (U.L.C.I.)
3. Find the 15th term of the A.P. whose first term is 463 and the 28th term is 4.
4. In 1924 a man was engaged at a certain salary rising by annual increments of £24 to a maximum of £556 per year which he received during 1939. Find (i) his commencing salary and (ii) his salary in 1930.
5. For the arithmetical series in which the first term is 91 and the 11th term is 21, find (i) the 27th term and (ii) the sum of 27 terms. Explain the result of (ii).
6. A clerk is engaged at a salary of £100 per annum and each year his salary rises by £12 10s. per annum. What will be his total earnings in 17 years? (U.L.C.I.)
7. A tradesman's takings during the first and last weeks of a certain quarter were £83 and £104 respectively. Assuming that the weekly takings form an arithmetical series, find (i) the increase

per week, (ii) the total sum taken during the period and (iii) the average weekly takings,

8. An article priced at £9 may be bought by paying a deposit of £1 at the time of the purchase together with five monthly payments of £1 15s. Calculate the rate per cent. per annum of simple interest charged.

9. A sum of £52 is borrowed on the condition that it is repaid in eight monthly payments of £7, the first to be paid one month after the loan is received. Calculate the rate per cent. per annum of simple interest charged.

10. Bicycles priced at £5 15s. can be purchased by a deposit of five shillings followed by eleven instalments of 10s. 9d. payable monthly, the first to be paid one month after purchase. Calculate the rate per cent. per annum of simple interest charged.

11. The price of a motor-car is £355. A purchaser agrees to pay a deposit of 25 per cent. and the balance, increased by £14 for interest, in 12 equal monthly instalments. Reckoning simple interest, what is the rate of interest charged? (R.S.A.)

12. A wardrobe, priced at 21 guineas, is purchased on the instalment plan. A deposit of £5 is made at the time of purchase and the balance, with interest, is cleared by the payment of 72 weekly instalments of 5s., the first being due one week after purchase. Calculate, correct to one place of decimals, the rate per cent. per annum of simple interest charged.

13. An advertisement offers an article for £50, and states that deferred payments on easy terms may be arranged. On enquiry, the deferred terms are found to be that £5 may be paid on purchase and thereafter twelve monthly instalments of £4 each. What is the rate of simple interest charged for the deferred terms? (C.I.S.)

14. A moneylender makes an advance of £30 on the condition that it is repaid by 15 monthly instalments, each of £2 10s., the first to be paid one month after the advance is made. Calculate the rate per cent. per annum of simple interest charged by the moneylender.

15. A college sessional fee of 34 guineas may be paid in three instalments of 12 guineas, each payable at the beginning of the term. Taking a term as three months, find the rate per cent. per annum of simple interest charged.

16. The cash price of an article is £19. It may be bought on the instalment plan, by which 10 payments of £2 each are made at intervals of one month, the first payment being made immediately. Find the rate of interest per annum, to the nearest tenth, on the basis of simple interest, which is charged under the instalment plan. (R.S.A.)

17. Certain stores offer goods on the following terms: 10 per cent. of the cash value to be paid down and the remainder, increased by $2\frac{1}{2}$ per cent. of its value, to be paid in twelve equal monthly instalments, the first instalment one month after purchase. Find the rate of simple interest per annum that the purchaser is really charged for the accommodation. (R.S.A.)

18. A catalogue states that an article may be purchased for £5 cash or, with $2\frac{1}{2}$ per cent. added to this price, by ten equal monthly payments, the first to be made at the time of purchase. Calculate the actual rate per cent. per annum of simple interest charged under this arrangement.

19. A man, desiring to buy a wireless receiving set costing £25, requests the vendor to allow him to spread the payment over one year in monthly instalments. The vendor replies that he "must raise the price to £26, thus charging 4 per cent. interest", and it is agreed that payment shall be made in 13 monthly instalments of £2 each, the first instalment to be paid immediately. Calculate, as a percentage per annum, the actual rate of interest by the vendor. (R.S.A.)

20. A medical school wishes to introduce an instalment plan of payment, and decides to charge simple interest at 5 per cent. per annum for this accommodation. If the fee is 81 guineas per session, calculate the instalment to be paid at the beginning of each of the three terms.

21. A man buys a car priced at £291 8s. He pays £95 down and agrees to settle the balance in twelve monthly instalments of £16 10s. each, the first to be paid one month after purchase. To the last instalment a certain sum is to be added in order to bring the simple interest on his debit balances up to the rate of 5 per cent. per annum. If this extra sum were spread evenly over the twelve instalments, how much extra must he pay each month?

CHAPTER XVI

THE GEOMETRICAL SERIES AND ITS PRACTICAL APPLICATION

16.1. Geometrical Progression.

WHEN the period of payment of a liability is spread over a long period, the interest necessarily becomes compound and the arithmetical series is no longer applicable. For example, the amounts at the ends of 1, 2, 3, ... years of £ P at r per cent. per annum compound interest are £ PR , £ PR^2 , £ PR^3 , ... , where £ R is the amount of £1 for one year.

Here then is a series which is not an A.P., since each succeeding term is formed by multiplying the preceding term by a constant number R . Such a series is called a **Geometrical Progression (G.P.)**.

If a denotes the first term of a G.P. and r the constant multiplier, then the general form of the series is

$$a, ar, ar^2, ar^3, \dots,$$

from which it will be evident that the

$$\text{2nd term} = ar,$$

$$\text{3rd } \quad \quad \quad = ar^2,$$

$$\text{4th } \quad \quad \quad = ar^3,$$

and so on, in which the power of r is just *one less than the number of the term* ; hence, the

$$\text{nth term} = ar^{n-1}.$$

The constant multiplier r is known as the **common ratio**, because it gives the ratio of any term to that preceding it.

Ex. 1. Find (i) the 5th term of the series, 16, 20, 25, ... , (ii) the 3rd term of the G.P. in which the first and sixth terms are 125 and 9·72 respectively.

In each of these examples, it is necessary first to find the common ratio r .

(i) In this series, $\frac{20}{16} = \frac{25}{20} = \dots = \frac{5}{4}$, so that $r = \frac{5}{4}$.

$$\therefore \text{the 5th term} = ar^4 = 16 \times \left(\frac{5}{4}\right)^4 = \frac{625}{16} = 39\frac{1}{16}.$$

(ii) In this case, the first term is 125 and if r be the common ratio, the sixth term is $125r^5$,

$$\therefore 125r^5 = 9\cdot72,$$

$$\text{or } r^5 = \frac{9\cdot72}{125} = \frac{972}{12500} = \frac{3 \times 3 \times 3 \times 3 \times 3 \times 4}{5 \times 5 \times 5 \times 5 \times 5 \times 4} = \left(\frac{3}{5}\right)^5;$$

$$\therefore r = \frac{3}{5}.$$

$$\text{Hence, the 3rd term} = 125 \times \left(\frac{3}{5}\right)^2 = \frac{125 \times 9}{25} = 45.$$

16·2. Sum of a Series in G.P.

To simplify the numerical computation in many problems, it is convenient to be able to replace a series of numbers in G.P. by their sum. This removes the necessity of calculating each term separately, as will be seen later.

Let S_n denote the sum of the series $a, ar, ar^2, \dots, ar^{n-1}$, in which there are n terms, then

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1};$$

$$\therefore r \times S_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$S_n - r \times S_n = a - ar^n,$$

i.e.

$$S_n(1 - r) = a(1 - r^n);$$

$$\therefore S_n = \frac{a(1 - r^n)}{1 - r} \text{ or } \frac{a(r^n - 1)}{r - 1}.$$

according as r is less or greater than unity.

Ex. 2. Write down an expression for the sum of the series

$$1 + R + R^2 + \dots + R^5,$$

and use it to calculate the exact sum as a decimal when $R = 1.2$; hence, find the value of x from the relation

$$PR^6 = x(1 + R + R^2 + \dots + R^5)$$

when $R = 1.2$ and $P = 6206.2$.

The series $1, R, R^2, \dots, R^5$ is a G.P. of six terms whose common ratio is R , and since the first term is 1,

$$\therefore 1 + R + R^2 + \dots + R^5 = \frac{R^6 - 1}{R - 1}.$$

Since $R = 1.2$,

$$\therefore R^6 = (1.2)^3 \times (1.2)^3 = 1.728 \times 1.728 = 2.985984.$$

$$\text{Hence, } \frac{R^6 - 1}{R - 1} = \frac{1.985984}{0.2} = 9.92992.$$

Substituting in the given equation,

$$6206.2 \times 2.985984 = x \times 9.92992 ;$$

$$\therefore x = \frac{6206.2 \times 2.985984}{9.92992} = \frac{6206.2 \times 2.985984}{32 \times 0.31031} = 1866.24.$$

Although the numbers here have been chosen so that the calculations may be made without tables, in most cases of money problems, it will generally be necessary to use seven-figure logarithms to obtain results correct to the nearest penny. Examples of such problems will next be discussed.

16.3. Repayment of Long Period Loans.

Suppose a loan be granted for a long period, such, for instance, as an advance from a Building Society for the purchase of property, then compound interest must be reckoned. In transactions of this type, where equal instalments have to be paid at regular intervals, the application of the compound interest formula involves the geometrical series explained above.

Ex. 3. *A man buys a house for £1368. He pays a deposit of £100 at once and borrows the balance at $4\frac{1}{2}$ per cent. per annum compound interest on the condition that the loan, with interest, should be repaid in five equal yearly instalments, the first to be due one year after the date of purchase. Calculate, to the nearest penny, the amount of each instalment.*

The actual loan is $\text{£}1368 - \text{£}100 = \text{£}1268$.

At the end of the 1st year this will amount to $\text{£}(1268 \times 1.045)$, hence, if $\text{£}x$ be the value of each instalment, the loan for the 2nd year will be $\text{£}(1268 \times 1.045 - x)$.

At the end of the 2nd year this amounts to

$$\text{£}\{(1268 \times 1.045 - x)\} \times 1.045 = \text{£}\{1268 \times (1.045)^2 - x \times 1.045\}.$$

Another instalment is now paid, so that the loan for the 3rd year is $\text{£}\{1268 \times (1.045)^2 - x \times 1.045 - x\}$, and, at the end of the 3rd year this amounts to

$$\begin{aligned} & \text{£}\{1268 \times (1.045)^2 - x \times 1.045 - x\} \times 1.045 \\ &= \text{£}\{1268 \times (1.045)^3 - x \times (1.045)^2 - x \times 1.045\}. \end{aligned}$$

Proceeding in this way, the amount to be paid at the end of the 5th year will become

$$\begin{aligned} & \text{£}\{1268 \times (1.045)^5 - x \times (1.045)^4 - x \times (1.045)^3 - x \times (1.045)^2 - x\} \\ & \quad \times 1.045, \end{aligned}$$

and this must evidently be equal to the final payment of $\text{£}x$, i.e.

$$\begin{aligned} & 1268 \times (1.045)^5 - x \times (1.045)^4 - x \times (1.045)^3 - x \times (1.045)^2 - x \\ & \quad \times 1.045 = x, \end{aligned}$$

or, putting the terms in x on the left-hand side :

$$x \times \{1 + 1.045 + (1.045)^2 + (1.045)^3 + (1.045)^4\} = 1268 \times (1.045)^5.$$

Now the series by which x has to be multiplied is a G.P. of five terms whose common ratio is 1.045 and first term 1 ; hence, its sum is

$$\frac{(1.045)^5 - 1}{1.045 - 1} = \frac{(1.045)^5 - 1}{0.045}.$$

The numerator of the fraction must first be evaluated ; from the tables,

$$\log 1\cdot045 = 0\cdot0191163,$$

$$5 \times \log 1\cdot045 = 0\cdot0955815 = \log (1\cdot246 + \epsilon).$$

By the method of proportional differences, $\epsilon = 0\cdot00018$.

Hence, $(1\cdot045)^5 = 1\cdot24618$, and $(1\cdot045)^5 - 1 = 0\cdot24618$.

$$\therefore x \times \frac{0\cdot24618}{0\cdot045} = 1268 \times (1\cdot045)^5,$$

$$\text{so that} \quad x = \frac{1268 \times (1\cdot045)^5 \times 0\cdot045}{0\cdot24618}.$$

Taking logarithms :

$$\log 1268 = 3\cdot1031193$$

$$5 \times \log 1\cdot045 = 0\cdot0955815$$

$$\log 0\cdot045 = \bar{2}\cdot6532125$$

$$\hline 1\cdot8519133$$

$$\log 0\cdot24618 = \bar{1}\cdot3912528$$

$$\hline 2\cdot4606605 = \log 288\cdot8421 ;$$

$$\therefore x = 288\cdot8421.$$

Hence, each instalment = £288·8421 = £288 16s. 10d.

Note that, if the equation

$$x \times \{1 + 1\cdot045 + (1\cdot045)^2 + (1\cdot045)^3 + (1\cdot045)^4\} = 1268 \times (1\cdot045)^5$$

be divided out by $(1\cdot045)^4$, then, writing k for $1/1\cdot045$, it becomes

$$x \times \{k^4 + k^3 + k^2 + k + 1\} = 1268 \times 1\cdot045,$$

and the series multiplying x is now a G.P. of five terms written in the reverse order, whose first term is 1 and common ratio k ; the sum of the series is therefore $(1 - k^5)/(1 - k)$, so that

$$x = \frac{1268 \times 1\cdot045 \times (1 - k)}{1 - k^5} = \frac{1268 \times 0\cdot045}{1 - k^5}.$$

Before logarithms are applied to evaluate this fraction, the value of $1 - k^5$ must be found.

$$\begin{aligned}\text{Now } \log k^5 &= 5 \times \log k = 5 \times (\log 1 - \log 1.045) \\ &= 0 - 0.0955815 = \bar{1}.9044185 = \log 0.802451 ;\end{aligned}$$

$$\therefore 1 - k^5 = 0.19755.$$

Hence

$$x = \frac{1268 \times 0.045}{0.19755}.$$

From the tables :

$$\begin{array}{rcl}\log 1268 & = & 3.1031193 \\ \log 0.045 & = & \bar{2}.6532125 \\ & & 1.7563318 \\ \log 0.19755 & = & \bar{1}.2956770 \\ & & 2.4606548 = \log 288.8383 \dots ,\end{array}$$

so that

$$x = 288.8383,$$

and each instalment = £288.8383 = £288 16s. 9d.

Thus there is a variation of nearly one penny, actually £0.0038 or 0.912 pence, between the two results. This is due to the fact that, whereas 1.045 is exact, $1/1.045$ does not yield an exact decimal, so that there will be a greater error in any power of $1/1.045$ than in the corresponding power of 1.045, the greater the power the greater being the error; hence, since 1.045 represents the particular value of R in the above example, the method of using the reciprocal of R instead of R itself is not, in general, likely to give so accurate a result, and is therefore not recommended.

16.4. A General Formula.

It is quite simple to derive a general formula from which the equal instalments may be directly calculated.

Suppose a loan of £ P at r per cent. per annum compound interest were to be repaid in n equal yearly instalments of £ x , the first to be due one year after the loan had been received, then, proceeding exactly as in Ex. 3,

$$PR^n - x(1 + R + R^2 + \dots + R^{n-1}) = 0, \text{ where } R = 1 + \frac{r}{100}.$$

But $1 + R + R^2 + \dots + R^{n-1} = \frac{R^n - 1}{R - 1}$, since the series in R is a G.P. of n terms whose common ratio is R .

$$x \times \frac{R^n - 1}{R - 1} = PR^n,$$

or
$$x = \frac{PR^n(R - 1)}{R^n - 1}.$$

Ex. 4. *A sum of £30,000 is borrowed for public works by a Town Council, the principal and compound interest at $3\frac{1}{2}$ per cent. per annum to be repaid by the end of 25 years by equal annual instalments commencing one year after the money is borrowed. Find the amount of each instalment to the nearest shilling.*

Here $P = 30,000$, $n = 25$, $r = 3\frac{1}{2}$, and therefore $R = 1.035$.

Hence, from the above formula :

$$x = \frac{30000 \times (1.035)^{25} \times 0.035}{(1.035)^{25} - 1}.$$

First, the value of the denominator must be determined.

Now $\log 1.035 = 0.0149403$;

$$\therefore 25 \times \log 1.035 = 0.3735075 = \log 2.3632,$$

so that
$$x = \frac{30000 \times (1.035)^{25} \times 0.035}{1.3632}.$$

Taking logarithms :

$$\begin{array}{rcl} \log 30,000 & = & 4.4771213 \\ 25 \times \log 1.035 & = & 0.3735075 \\ \log 0.035 & = & \bar{2}.5440680 \\ & & 3.3946968 \\ \log 1.3632 & = & 0.1345596 \\ & & \hline & & 3.2601372 = \log 1820.275\dots, \end{array}$$

only three places being needed, as the result is required to the nearest shilling.

$$\therefore x = 1820.275,$$

and each instalment = £1820.275 = £1820 6s.

16·5. Sinking Funds.

When a company has to redeem certain liabilities, such as debentures (*see* Section 9·2, page 143) at some future specified date, a certain sum is generally invested each year at compound interest so that it will produce the requisite amount on the given date. A fund created in this way is known as a **sinking fund**.

Suppose the sum invested at the end of each year is $\pounds P$, then at r per cent. per annum this becomes $\pounds PR$ at the end of the second year, where R as usual denotes the amount of $\pounds 1$ for one year at r per cent. Another $\pounds P$ is now invested so that the sinking fund becomes $\pounds (PR + P)$ or $\pounds P(R + 1)$.

Similarly, at the end of the third year, the fund becomes

$$\pounds \{P(R + 1)R + P, \text{ or } \pounds P(R^2 + R + 1),$$

and so on.

Hence, if the amount is $\pounds S$ at the end of n years,

$$S = P(1 + R + R^2 + \dots + R^{n-1}).$$

But $1 + R + R^2 + \dots + R^{n-1} = \frac{R^n - 1}{R - 1}$, from Section 16·4.

Therefore
$$S = \frac{P(R^n - 1)}{R - 1},$$

where $\pounds S$ is the amount of the sinking fund and $\pounds P$ is the sum invested at the end of each year for n years.

Ex. 5. *A firm has to redeem a liability of $\pounds 19,000$ at the end of 25 years and creates a sinking fund by investing a certain sum at the end of each year at $3\frac{1}{2}$ per cent. per annum compound interest. Find this sum to the nearest \pounds .*

Here $S = 19,000$, $R = 1 + 0\cdot035$, $n = 25$ and P has to be calculated.

$$\therefore 19000 = \frac{P\{(1\cdot035)^{25} - 1\}}{1\cdot035 - 1},$$

so that

$$P = \frac{19000 \times 0\cdot035}{(1\cdot035)^{25} - 1}.$$

Now $\log 1·035 = 0·0149403$;

$$\therefore 25 \cdot \log 1·035 = 0·3735075 = \log 2·363238,$$

and $(1·035)^{25} - 1 = 2·363238 - 1 = 1·363238$.

$$\text{Hence, } P = \frac{19000 \times 0·035}{1·363238} = \frac{665}{1·363238} = 487·7 \dots$$

\therefore To the nearest £, the sum to be invested at the end of each year = £488.

16·6. Determination of the Time Period.

When it is necessary to estimate how long it will take to redeem a debt, the value of n must be determined. As explained in Section 14·6, page 222, tables must be used, since n is an index and four-figure logarithms will generally suffice when n is a number of years less than 100.

Ex. 6. *A company borrows £7,000 at $4\frac{1}{2}$ per cent. per annum compound interest and agrees to repay the debt with interest in equal annual instalments, each of £495, the first to be paid one year from the date of borrowing. How many years will the complete repayment take?*

Let n be the number of years, then the sum to be repaid is $£7000 \times (1·045)^n$, and from the formula of Section 16·4,

$$\begin{aligned} 7000 \times (1·045)^n &= \frac{495 \{(1·045)^n - 1\}}{1·045 - 1} = \frac{495 \{(1·045)^n - 1\}}{0·045} \\ &= 11000 \{(1·045)^n - 1\}. \end{aligned}$$

Divide throughout by 1000, then

$$7 \times (1·045)^n = 11 \times (1·045)^n - 11,$$

$$\text{i.e. } 4 \times (1·045)^n = 11,$$

$$\text{or } (1·045)^n = \frac{11}{4} = 2·75.$$

Hence, taking logarithms, $n \times \log 1·045 = \log 2·75$,
so that $n \times 0·0191 = 0·4393$;

$$\therefore n = \frac{0·4393}{0·0191} = \frac{4393}{191} = 23.$$

Hence, the complete repayment will take **23** years.

EXERCISES 16

1. Find the sum to 8 terms of the geometrical progression : $\frac{4}{7}$, $\frac{3}{7}$, $\frac{9}{28}$, (U.L.C.I.)

2. The population of a town was 365,428 in 1901, 439,611 in 1911 and 528,849 in 1921. Shew that the population was increasing approximately in G.P. Assuming the same approximate rate of increase, calculate the probable population in 1941.

3. Find, by use of logarithms, the eighth term of the geometrical progression 1, 2.9, 8.41, (U.L.C.I.)

4. A certain sum of money invested at compound interest amounts to £8400 in two years and £9261 in four years. Calculate (i) the rate of interest per cent. per annum and (ii) the sum invested, to the nearest £.

5. Find the sum of seven terms of the geometric series $1\frac{1}{29}$, $\frac{20}{29}$, $\frac{40}{87}$,

Find also, by means of tables, which term of the above series is the first to be less than 0.001. (U.L.C.I.)

6. A man left a sum of money to be divided amongst his four children *A*, *B*, *C*, *D*, in G.P. *B* received £900 and *D* £729. Find the total sum left.

7. A series of numbers are in G.P. The sum of the first and third terms is $108\frac{3}{4}$ and the sum of the second and fourth terms is $43\frac{1}{2}$. Find the fifth term.

8. A sum of £30,000 is borrowed for public works by a Town Council, the principal and interest at $4\frac{1}{2}$ per cent. compound interest per annum to be paid off by the end of 17 years, by equal annual instalments commencing one year after the money is borrowed. Find the value of each instalment. (R.S.A.)

9. A sum of £40,000 is borrowed at $4\frac{1}{2}$ per cent. per annum compound interest. It has to be repaid in 25 equal yearly instalments, the first being due one year after the loan was received. Calculate, to the nearest penny, the amount of each instalment.

10. A Town Council borrows £20,000 at $3\frac{1}{2}$ per cent. per annum. What sum must the Council raise yearly in order to pay the

interest on the loan regularly and to establish a sinking fund, the money to accumulate at 3 per cent. per annum compound interest to pay off the capital of the loan in 20 years? (R.S.A.)

11. A company borrows £10,000 at 3 per cent. per annum compound interest reckoned half-yearly. It is arranged that the payment of the debt with interest shall be made in 24 half-yearly instalments, the first 23 instalments being £500 each. Calculate, to the nearest £, the amount of the 24th instalment to be paid at the end of 12 years. (R.S.A.)

12. The price of a piano is 150 guineas cash, or it may be paid for by 36 equal monthly instalments, the first instalment to be paid at the time of purchase. Reckoning 5 per cent. compound interest, find what the amount of this monthly instalment should be. (R.S.A.)

13. A machine costing £1000 is to be paid for in three equal instalments, the first being paid at the end of the first year. Find, on the basis of compound interest at 5 per cent. per annum, the amount of each instalment to the degree of accuracy permitted by the tables. (R.S.A.)

14. A man buys a leasehold house for £875, the lease having 35 years still to run. Reckoning compound interest at 3 per cent., find what annual payment to an insurance company he must make so as to get back the price of the house at the expiration of the lease. The first payment to the insurance company to be made at the time the house is bought. (R.S.A.)

15. A man, having just bought a leasehold house, the lease having 23 years to run, arranges to get back the purchase price, £950, by 23 annual payments to an insurance company, the first payment to be made at once. Reckoning compound interest at $3\frac{1}{2}$ per cent. per annum, what should each payment be? (R.S.A.)

16. A loan of £56,000, bearing interest at $4\frac{1}{2}$ per cent. per annum, has to be paid off in twenty years by 20 equal annual instalments. Find, to the nearest £, the value of each instalment. (U.L.C.I.)

17. To reduce a debt of £15,000, £1500 was paid at the end of one year from borrowing and at the end of each successive year until ten such payments had been made. The annual payment at the end of each of the following years was reduced to £1000. How

much was owing after 15 payments in all had been made, reckoning compound interest at 5 per cent. per annum? (L.Ch.C.)

18. A man borrowed £500, agreeing to pay back the debt with interest in ten equal half-yearly instalments, the first instalment to be paid at the end of six months. On the basis of $3\frac{1}{2}$ per cent. compound interest, reckoned half-yearly, calculate the value of each instalment. (R.S.A.)

19. A sum of £230 is invested on the first day of 1927 and an equal sum on the first days of 1928 and 1929. Find the amount at the end of 1929, allowing compound interest at $4\frac{1}{2}$ per cent. per annum convertible yearly. Calculate also the single sum which, invested on 1st January, 1927, would at the same rate reach the same amount as the previous answer at the end of 1929. (C.I.S.)

20. A loan of £25,000, together with compound interest thereon at the rate of 5 per cent. per annum, is paid off in six equal annual instalments, each carrying compound interest at 5 per cent. per annum, the first payment being made at the end of the first year. Find the amount of each payment. (C.I.S.)

21. A company borrows £7000 and agrees to repay the debt with interest in equal annual instalments of £600, the first to be paid one year after the money is borrowed. How many years will it take to liquidate the debt, reckoning compound interest at 4 per cent. per annum?

22. A company incurred a debt of £40,000 for plant and agreed to discharge it by annual payments, the first to be made one year after installation. The first ten payments were £2000 each and the succeeding payments £4000 each. After how many years would the debt be discharged, reckoning compound interest at 5 per cent. per annum? (L.Ch.C.)

23. A sum of £4600 is borrowed at $3\frac{1}{2}$ per cent. per annum compound interest subject to the agreement that the debt and interest must be paid back in equal yearly instalments, each of £336, the first to be paid one year from the date of the loan. Calculate how many years it will take to repay the debt completely.

24. A factory management installed some new machinery costing £2950. £500 was paid down and the balance with compound interest at $3\frac{1}{2}$ per cent. per annum, in ten equal yearly instalments,

the first to be made one year after purchase. Calculate, to the nearest £, each instalment.

25. To redeem certain liabilities, a company decides to create a sinking fund of £10,000 at the end of 21 years by investing a constant sum at the beginning of each of these years. Reckoning compound interest at $3\frac{1}{2}$ per cent. per annum, calculate, to the nearest penny, what the annual investment must be.

26. Find, approximately, how many years it will take to create a sinking fund which will be 25 times the annual investment made, reckoning compound interest at $4\frac{1}{2}$ per cent. per annum.

CHAPTER XVII

ANNUITIES

17.1. Definitions.

A SERIES of payments made at regular intervals during a period of time is known as an **Annuity**.

To obtain such a series of payments, it is necessary to invest an adequate sum of money—called the **Purchase Price**—which, together with the interest earned, is repayable as an annuity for a given time, this period and the amount of each payment depending upon the actual amount invested.

An annuity which is payable for a fixed number of years is usually called an **annuity-certain**. Such an annuity is said to be :

- (i) *Due* when the first payment is to be made at the *beginning* of the first year or other agreed interval.
- (ii) *Immediate* when the first payment is to be made at the *end* of the first year or other agreed interval.
- (iii) *Deferred* when the first payment is to be made *after an agreed number of years* or other intervals have elapsed.

When the payments of an annuity, such as the rent of a freehold estate, are to continue for ever, the annuity is known as a **Perpetuity**.

17.2. Case of an Immediate Annuity-Certain.

Suppose an annuity of $\text{£}P$ per annum for n years can be purchased for $\text{£}A$, the first payment to be made one year after purchase and compound interest being reckoned at r per cent. per annum ; then, writing R for $1 + \frac{r}{100}$ as usual, $\text{£}A$ will become $\text{£}AR$ when the first payment becomes due. Hence, after $\text{£}P$ has been paid, the sum remaining is $\text{£}(AR - P)$ which, at the end of the second year becomes $\text{£}(AR - P) \times R$ or $\text{£}(AR^2 - PR)$. Another

In order to facilitate calculation, it may be convenient in some cases to use the following form of the above formula :

$$\frac{A}{P} = \frac{1 - \frac{1}{R^n}}{R - 1}.$$

Ex. 1. *The purchase price of an annuity is £2150 and it is to consist of 16 equal annual instalments, the first to be payable one year after purchase. Reckoning compound interest at 3 per cent. per annum, calculate, to the nearest penny, the value of each instalment.*

Evidently this is a case of an Immediate Annuity-certain, so that the formula already established may be directly applied.

Putting $A = 2150$, $R = 1.03$ and $n = 16$;

$$\frac{2150}{P} = \frac{(1.03)^{16} - 1}{(1.03)^{16} \times 0.03},$$

or
$$P = \frac{2150 \times (1.03)^{16} \times 0.03}{(1.03)^{16} - 1}.$$

The value of $(1.03)^{16}$ must first be determined :

Now $\log 1.03 = 0.0128372$;

$\therefore 16 \log 1.03 = 0.2053952 = \log 1.6047$ approximately.

Hence,
$$P = \frac{2150 \times 1.6047 \times 0.03}{0.6047}.$$

Taking logarithms :

$$\log 2150 = 3.3324385$$

$$\log 1.6047 = 0.2053952$$

$$\log 0.03 = \bar{2}.4771213$$

$$\hline 2.0149550$$

$$\log 0.6047 = \bar{1}.7815400$$

$$\hline 2.234150 = \log 171.1650.$$

\therefore Each instalment = £171.1650

= £171 3s. 4d. to the nearest penny.

By the use of the alternative form of the formula, the calculation would appear as follows :

$$\frac{2150}{P} = \frac{1 - \frac{1}{(1.03)^{16}}}{0.03}.$$

$$\text{Now} \quad \log \frac{1}{(1.03)^{16}} = \log 1 - 16 \log 1.03 = -0.2053952.$$

To render the decimal part of this logarithm positive, write 0 as +1-1; then

$$-0.2053952 = -1 + 1 - 0.2053952 = \bar{1}.7946048 = \log 0.62317.$$

$$\text{Hence,} \quad P = \frac{2150 \times 0.03}{1 - 0.62317} = \frac{64.5}{0.37683} = 171.1647 \dots$$

Thus, as before, each instalment = £171.1647... = £171 3s. 4d.

Note that in the final calculation for P no logarithms need be used.

Ex. 2. *A man buys an annuity of £104 per year for 15 years, the first payment to be made one year after purchase. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate the purchase price to the nearest £.*

This is also a case of an immediate annuity-certain; hence the formula of Section 17·2 may be applied.

Taking $P=104$, $R=1.025$ and $n=15$:

$$A = \frac{104 \times \{(1.025)^{15} - 1\}}{(1.025)^{15} \times 0.025}.$$

To find the value of $(1.025)^{15}$ first :

$$\log 1.025 = 0.0107239 ;$$

$$\therefore 15 \log 1.025 = 0.1608585 = \log 1.4483.$$

Hence

$$\begin{aligned} A &= \frac{104 \times 0.4483}{(1.025)^{15} \times 0.025} \\ &= \frac{104 \times 0.4483}{1.4483 \times 0.025}. \end{aligned}$$

Taking logarithms :

$$\begin{array}{rcl}
 \log 104 & = & 2.0170333 \\
 \log 0.4483 & = & \bar{1}.6515687 \\
 & & \underline{1.6686020} \\
 & & \bar{2}.5587985 \leftarrow \\
 & & \underline{3.1098035} = \log 1287.66\dots
 \end{array}$$

$\log 1.4483 = 0.1608585$
 $\log 0.025 = \bar{2}.3979400$
 $\underline{\bar{2}.5587985}$

Hence, the cost of the annuity, to the nearest £ = £1288.

As a check, the alternative formula gives :

$$\begin{aligned}
 A &= \frac{104 \times \{1 - 1/(1.025)^{15}\}}{0.025} = 4160 \times (1 - 0.690464) \\
 &= 4160 \times 0.309536 = 1287.67,
 \end{aligned}$$

i.e. to the nearest £, the purchase price is £1288.

17.3. Due and Deferred Annuities-Certain.

In the case of a due annuity-certain, the first payment is made immediately after purchase so that, if the annuity is to continue for n years, there will in general be $(n+1)$ equal payments ; hence, by a slight adaptation of formula (i) of Section 17.2,

$$\frac{A}{P} = \frac{R^{n+1} - 1}{R^n(R - 1)} \dots\dots\dots (ii)$$

Again, for a deferred annuity of n equal annual payments, suppose the first payment is to be made m years after purchase, then the purchase price of £ A becomes £ AR^m at the end of m years ; hence, from formula (ii), since there are only n payments,

$$\frac{AR^m}{P} = \frac{R^n - 1}{R^{n-1}(R - 1)},$$

$$\text{i.e.} \quad \frac{A}{P} = \frac{R^n - 1}{R^{m+n-1}(R - 1)} \dots\dots\dots (iii)$$

Note that, for an immediate annuity-certain, $m=1$ and formula (iii) reduces to (i).

Ex. 3. *On his 43rd birthday, a man wishes to buy an annuity of £220 per year, the number of annual payments to be 16 and the first*

to be made on his 60th birthday. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate the purchase price to the nearest penny.

Here the first payment is to be made 60-43 or 17 years after purchase; hence, in formula (iii), $m=17$, $n=16$, $P=220$ and $R=1.025$:

$$\therefore \frac{A}{220} = \frac{(1.025)^{16} - 1}{(1.025)^{32} \times 0.025}.$$

The value of $(1.025)^{16}$ must first be determined before A can be calculated.

$$\text{Now } \log 1.025 = 0.0107239,$$

$$16 \cdot \log 1.025 = 0.1715824 = \log 1.48451 \text{ approximately.}$$

$$\text{Hence } A = \frac{220 \times 0.48451}{(1.025)^{32} \times 0.025}.$$

Taking logarithms:

$\log 220 = 2.3424227$	$32 \log 1.025 = 0.3431648$
$\log 0.48451 = 1.6853027$	$\log 0.025 = 2.3979400$
2.0277254	2.7411048
2.7411048	
$3.2866206 = \log 1934.7310 \dots$	

$$\therefore \text{Purchase price of the annuity} = \text{£}1934.7310$$

$$= \text{£}1934 \text{ 14s. 7d. to the nearest penny.}$$

The price has been calculated to the nearest penny for practice, but generally the nearest £ is sufficient. To the nearest £ the price would obviously be £1935.

17.4. Purchase Price of a Deferred Annuity in Instalments.

Instead of paying down a lump sum for a deferred annuity, it is often possible to purchase it by making a number of regular equal payments spread over a period, as for instance, through an Insurance Company. Each payment is then called a **Premium**.

Suppose an annuity of £ P per annum be bought by m payments of £ K , each paid at the beginning of the year and the first payment of the annuity to become due one year after the last premium is paid, i.e. at the beginning of the $(m+1)$ th year.

The actual number of annuity payments is usually reckoned on the **expectation of life** of the annuitant. This is the number of years he is expected to live after receiving the first payment of the annuity. It therefore determines the number of annuity payments to be made; let this number be n , then, the present value of the annuity at the date of its first payment is, in £s,

$$\begin{aligned} & P + \frac{P}{R} + \frac{P}{R^2} + \dots + \frac{P}{R^{n-1}} \\ &= P \left(1 + \frac{1}{R} + \frac{1}{R^2} + \dots + \frac{1}{R^{n-1}} \right) \\ &= \frac{P(1 - 1/R^n)}{1 - 1/R} = \frac{P(R^n - 1)}{R^{n-1}(R - 1)}. \end{aligned}$$

But the value of the total premiums paid, on the same date, is in £s,

$$\begin{aligned} & KR^{m+1} + KR^m + \dots + KR \\ &= KR(R^m + R^{m-1} + \dots + 1) \\ &= \frac{KR(R^{m+1} - 1)}{R - 1}. \end{aligned}$$

Hence,
$$\frac{KR(R^{m+1} - 1)}{R - 1} = \frac{P(R^n - 1)}{R^{n-1}(R - 1)},$$

from which
$$\frac{K}{P} = \frac{R^n - 1}{R^n(R^{m+1} - 1)}. \dots\dots\dots(\text{iv})$$

As in the case of (i), it will sometimes render particular calculations less cumbersome if the above formula is applied in the form :

$$\frac{K}{P} = \frac{1 - \frac{1}{R^n}}{R^{m+1} - 1},$$

though this modification does not simplify the numerical work so much as in the similar modification of (i).

Ex. 4. *In order to make provision for retirement at the age of 60, a man decides to buy an annuity of £120 per annum by paying an Insurance Company a yearly premium. Calculate, to the nearest penny, what that premium should be if the first is to be paid on his 40th birthday and the last on his sixtieth birthday, assuming that his expectation of life is 15 years from the date he is 61, on which date he is to receive the first payment of the annuity. Reckon compound interest at $2\frac{1}{4}$ per cent. per annum and take $\log 1.0225 = 0.0096633$.*

Since the number of birthdays from the 40th to the 60th is 21 inclusive, therefore 21 premiums must be paid, so that $m = 21$.

Also, since the man's expectation of life at the age of 61 is 15 years, $n = 15$; hence, since $P = 120$ and $R = 1.0225$, from formula (iv),

$$\frac{K}{120} = \frac{(1.0225)^{15} - 1}{(1.0225)^{15} \times \{(1.0225)^{22} - 1\}}.$$

To determine first the approximate values of $(1.0225)^{15}$ and $(1.0225)^{22}$;

$$\log 1.0225 = 0.0096633 ;$$

$$\therefore 15 \log 1.0225 = 0.1449495 = \log 1.39621,$$

and $22 \log 1.0225 = 0.2125926 = \log 1.63152.$

$$\text{Hence } K = \frac{120 \times 0.39621}{(1.0225)^{15} \times 0.63152} = \frac{120 \times 0.39621}{1.39621 \times 0.63152}.$$

Using logarithms :

$\begin{array}{r} \log 120 \quad = 2.0791812 \\ \log 0.39621 = \bar{1}.5979254 \\ \hline 1.6771066 \\ \bar{1}.9453366 \leftarrow \\ \hline 1.7317700 = \log 53.9225 \dots \end{array}$	$\begin{array}{r} \log 1.39621 = 0.1449495 \\ \log 0.63152 = \bar{1}.8003871 \\ \hline \bar{1}.9453366 \end{array}$
--	---

Hence the annual premium is £53.9225

$$= \text{£}53 \text{ 18s. 5d.}$$

Using the alternative form of formula (iv) :

$$\begin{aligned}\frac{K}{120} &= \frac{1 - 1/(1.0225)^{15}}{(1.0225)^{22} - 1} \\ &= \frac{1 - 0.7162267}{0.63152} = \frac{0.2837733}{0.63152} \\ \therefore K &= \frac{120 \times 28377.33}{63152} = \frac{3405279.6}{63152} = 53.9219,\end{aligned}$$

and

$$£53.9219 = £53 \text{ 18s. 5d.}$$

Note that the difference between the decimal values obtained in the two calculations is $£53.9225 - £53.9219 = £0.0006$, which is less than a farthing.

17.5. Perpetual Annuities.

As already mentioned, perpetual annuities are mainly connected with the rents of estates. Suppose, for instance, that a rent brings in $£P$ per annum, then the P.V. of n years' rent would be, in $£s$,

$$\frac{P}{R} + \frac{P}{R^2} + \frac{P}{R^3} + \dots + \frac{P}{R^n},$$

compound interest at r per cent. per annum being reckoned and, as usual, $R = 1 + r/100$.

This P.V. becomes, on taking out the common factor, P/R :

$$\begin{aligned}&\frac{P}{R} \left(1 + \frac{1}{R} + \frac{1}{R^2} + \dots + \frac{1}{R^{n-1}} \right), \\ &= \frac{P(1 - 1/R^n)}{R(1 - 1/R)} = \frac{P(1 - 1/R^n)}{R - 1}.\end{aligned}$$

Now, whatever the interest reckoned, R will always be greater than unity, so that $1/R^n$ will be less than unity. As therefore n becomes greater, $1/R^n$ will become less ; hence, as n is indefinitely increased, $1/R^n$ becomes indefinitely small. Indeed, $1/R^n$ may be made as small as we please by making n sufficiently great. When n therefore tends to become infinitely great, $1/R^n$ tends to zero.

Hence, considering the rent to continue in perpetuity,

$$\frac{P(1 - 1/R^n)}{R - 1} \text{ becomes } \frac{P}{R - 1};$$

thus, the value, A , of a Perpetual Annuity is given by the formula

$$A = \frac{P}{R - 1} = \frac{100P}{r}, \dots\dots\dots(v)$$

since $R = 1 + \frac{r}{100}$.

The arithmetic in this case is therefore very simple.

Ex. 5. *The rent of an estate is £351 per annum. At what price should it be sold to yield a return of $3\frac{3}{4}$ per cent. per annum?*

Here $P = 351$ and $R = 1.0375$.

Hence, from formula (v),

$$A = \pounds \frac{351}{0.0375} = \pounds 9360.$$

17.6. Miscellaneous Problems.

In the exercises already solved in this chapter so far, the unit of time has been taken as one year, but, frequently in practice, payments are made either quarterly or half-yearly. The method of solution, however, is the same provided the proper value of R is taken. The following exercises will illustrate the necessary adaptation when the time interval is less than one year.

Ex. 6. *A man borrows £750 on the understanding that he will repay it, with interest, in six half-yearly instalments, the first to be paid six months after the date of borrowing. If he pays five instalments each of £140, how much must the sixth instalment be to clear the debt, reckoning compound interest at $3\frac{1}{2}$ per cent. per annum? Take $\log 1.0175 = 0.0075344$.*

Since the payments are to be made half-yearly, it will be convenient to take R as the amount of £1 for 6 months at $3\frac{1}{2}\%$ per annum, i.e. $R = 1.0175$.

Let $\pounds P$ be the final payment; then the total P.V. of the six payments is :

$$\begin{aligned} & \pounds \left(\frac{140}{R} + \frac{140}{R^2} + \frac{140}{R^3} + \frac{140}{R^4} + \frac{140}{R^5} + \frac{P}{R^6} \right); \\ \therefore 140 \left(\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4} + \frac{1}{R^5} \right) + \frac{P}{R^6} &= 750. \end{aligned}$$

Clear off fractions by multiplying throughout by R^6 , then

$$\begin{aligned} P &= 750R^6 - 140(R + R^2 + R^3 + R^4 + R^5) \\ &= 750R^6 - 140R(1 + R + R^2 + R^3 + R^4) \\ &= 750R^6 - \frac{140R(R^5 - 1)}{R - 1} \\ &= 750R^6 - \frac{140(R^6 - R)}{0.0175} \\ &= 750R^6 - 8000(R^6 - R). \end{aligned}$$

It is now necessary to find the value of R^6 , i.e. $(1.0175)^6$:

$$\begin{aligned} \log 1.0175 &= 0.0075344, \\ 6 \log 1.0175 &= 0.0452064 = \log 1.109702. \\ \therefore P &= 750 \times 1.109702 - 8000(1.109702 - 1.0175) \\ &= 832.2765 - 737.6160 = 94.6605. \end{aligned}$$

Hence, the final payment will be $\pounds 94.6605$

$= \pounds 94 \text{ 13s. 3d.}$ to the nearest penny.

Ex. 7. *A man left instructions that when he died his executors were to purchase for his daughter an annuity of $\pounds 234$ per annum, payable in equal quarterly instalments. Calculate the purchase price of the annuity, reckoning compound interest at 3 per cent. per annum. Assume that the daughter's expectation of life to be 16 years from the date of her father's death and that the first payment of the annuity to be made three months after purchase.*

It may be assumed that the date of purchase and the date of the father's death were the same, for purposes of calculation.

Taking R as the amount of £1 for 3 months at 3 per cent. per annum, $R = 1·0075$.

Now there will be 64 quarterly payments of £58 10s., so that, if £ A be the purchase price, then from formula (i) :

$$A = \frac{58·5 \times (R^{64} - 1)}{R^{64}(R - 1)}.$$

To find the value of R^{64} , i.e. $(1·0075)^{64}$,

$$\log 1·0075 = 0·0032451,$$

$$\therefore 64 \log 1·0075 = 0·2076864 = \log 1·61319.$$

$$\text{Hence, } A = \frac{58·5 \times 0·61319}{(0·0075)^{64} \times 0·0075} = \frac{58·5 \times 0·61319}{1·61319 \times 0·0075}.$$

Taking logarithms :

$$\log 58·5 = 1·7671559$$

$$\log 1·61319 = 0·2076864$$

$$\log 0·61319 = \bar{1}·7875951$$

$$\log 0·0075 = \bar{3}·8750613$$

$$\underline{\bar{1}·5547510}$$

$$\underline{\bar{2}·0827477}$$

$$\underline{\bar{2}·0827477}$$

$$3·4720033 = \log 2964·8544.$$

\therefore the purchase price is £2964·8544

= £2965 to the nearest £.

Ex. 8. *A man pays an annual premium of £40 8s. for 27 years in order to secure a yearly annuity of £130, the first payment of which is to fall due one year after the last premium is paid. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate the number of annuity payments to be made.*

Here formula (iv) is applicable.

Putting $K = 40·4$, $P = 130$, $m = 27$, $R = 1·025$ and using the alternative form of the formula :

$$\frac{40·4}{130} = \frac{1 - 1/(1·025)^n}{(1·025)^{28} - 1},$$

the only unknown being n .

$$\begin{aligned}
 \text{Hence, } 1 - \frac{1}{(1.025)^n} &= \frac{40.4 \times \{(1.025)^{28} - 1\}}{130} \\
 &= \frac{40.4 \times 0.9965}{130}, \text{ on applying logarithms,} \\
 &= 0.309682. \\
 \therefore \frac{1}{(1.025)^n} &= 1 - 0.309682 = 0.690318.
 \end{aligned}$$

Since n is an index and therefore a logarithm, tables must be used to find its value.

$$\begin{aligned}
 \text{Hence, } n \log 1.025 &= \log \frac{1}{0.690318} = 0 - \log 0.690318 \\
 &= 0 - \bar{1}.8390492 = 1 - 0.8390492,
 \end{aligned}$$

$$\text{i.e. } n \times (0.0107239) = 0.1609508 ;$$

$$\therefore n = \frac{1609508}{107239} = 15.0 \dots$$

Thus there should be **15 payments**.

This means that, from the date of the first payment of the annuity, the expectation of life of the annuitant is 14 years, or from the date of the last payment of the premium, it is 15 years.

EXERCISES 17

In the following exercises, answers in money should be given to the nearest penny unless otherwise stated.

1. An annuity is purchased for £1300 and is to consist of 16 equal annual payments, the first to become due one year after purchase. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate each of the annual payments.

2. A man has recently bought a leasehold house for £1350. He now agrees to make 43 equal annual payments to an insurance company, the first when the lease has 43 years still to run, in return for the repayment by the insurance company of the purchase price at the expiration of the lease. On the basis of compound interest at 3 per cent. per annum, calculate the annual payment to the nearest shilling. Take $\log 103 = 2.0128372$. (R.S.A.)

3. Find what annuity for 28 years can be purchased for £4000, the rate of interest being $3\frac{1}{2}$ per cent. per annum.

The necessary logarithms will be found among those given below.

$$\log 1.035 = 0.01494035,$$

$$\log 2.620172 = 0.4183298,$$

$$\log 3.816542 = 0.5816702.$$

(U.L.C.I.)

4. A gentleman wishes to provide enough money to clear off a mortgage of £2500 when the opportunity occurs in 12 years' time, by investing annually 12 equal amounts, the first of which is to be invested in a year's time at 4 per cent. per annum compound interest. Calculate, to the nearest shilling, what each of these amounts must be? (C.I.S.)

5. A man owns a leasehold house. The ground rent is £25 a year and the local rates 11s. 6d. in the £ on £90. By offering to go on paying the local rates as well as the ground rent, he gets a tenant who rents the house from him at £130 for the remaining 15 years of the lease. If he receives an offer to buy the house subject to the conditions on which the house is let, what could he fairly ask for it, reckoning compound interest at $3\frac{1}{2}$ per cent. per annum? (R.S.A.)

6. What sum of money must a man pay at the age of 30 in order to buy an annuity of £300 a year, the first payment of the annuity to be made when he is 50. Assume that he will receive 15 payments before he dies, and calculate 3 per cent. per annum compound interest. (R.S.A.)

7. On his 40th birthday a man took out an insurance policy for £1000 payable in twenty years or at earlier death. The annual premium was £41 4s. payable half-yearly. If he died on the day before he was 50, what was the loss to the company on payment of the policy, reckoning interest at 3 per cent. per annum? (L.Ch.C.)

8. At the age of 32 a man pays £1000 to an insurance company to buy an annuity, the first annual payment of the annuity to be made when he is 65. If the insurance company reckon at 3 per cent. per annum compound interest and on the assumption that probably ten annual payments of the annuity will be made, what will be the annual value of the annuity? (R.S.A.)

9. The rent of an estate is £221 per annum and it is sold for £5200. Find the rate per cent. per annum of the yield.

10. A man decides to invest £100 on each of his birthdays from the 31st to the 60th inclusive, so that he may then purchase a life annuity. What will be the value of the annuity if the expectation of life of a man of 60 is 15 years, reckoning interest at 3 per cent. per annum? (L.Ch.C.)

11. The purchase price of an annuity of £100 for 15 years is spread over thirty years in 30 equal premiums. The annuity is to be paid in 15 annual instalments of £100 each, the first instalment one year after the last premium. On the basis of compound interest at 3 per cent. per annum, calculate the annual premium. (R.S.A.)

12. What sum of money would a man need to invest on each of his birthdays from the 21st to the 50th inclusive, so that he might receive £500 on each birthday from the 51st to the 60th inclusive, reckoning compound interest at $3\frac{1}{2}$ per cent. per annum? (L.Ch.C.)

13. By paying a lump sum on his 59th birthday, a man arranged with an insurance company to pay him an annuity of £100 per annum for the rest of his life, the first payment to be made on his 60th birthday. The insurance company reckons at 3 per cent. per annum compound interest on the assumption that he will probably live until he is 73. What lump sum must he pay? (R.S.A.)

14. On his 55th birthday a man buys an annuity of £300 to be paid to him in twenty half-yearly instalments of £150 each, commencing on his 60th birthday. Calculate the purchase price on the basis of 3 per cent. per annum compound interest, reckoned half-yearly. (R.S.A.)

15. A local council offers loans for the purchase of house property, such loans to be repaid with compound interest at $3\frac{1}{2}$ per cent. per annum in equal half-yearly instalments spread over agreed periods. Calculate the half-yearly instalments for a loan of £250 spread over ten years, the first instalment to be paid six months after the date the loan is made.

16. What annuity, to continue for 13 years, can be purchased for £4000? It is to be paid in 26 equal half-yearly instalments, the first instalment one year after purchase. Give the annual value of the annuity to the nearest ten shillings, and work this question on the basis of compound interest at 3 per cent. per annum, reckoned half-yearly. (R.S.A.)

17. A man buys an annuity for £2500 which is to be paid in equal half-yearly instalments, the first to become due one year after the date of purchase. Reckoning 32 instalments and compound interest at 3 per cent. per annum, calculate the amount of each half-yearly instalment.

18. A man left instructions in his will that his executors should purchase for his daughter a life annuity of £260 to be paid in instalments of £65 each at the beginning of each quarter. When he died his daughter's expectation was 16 years. Calculate the purchase price of the annuity, to consist of 64 successive quarterly payments, on the basis of compound interest at 4 per cent. per annum, reckoned quarterly. (R.S.A.)

19. A Government annuity of £50 per year is purchased on October 5th, 1936, for £624. The annuity payments are made quarterly and begin on January 5th, 1937. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate the number of payments to be made. Take $\log 1006\cdot25 = 3\cdot0027059$.

20. A man, aged 60, by paying £1000 to an insurance company, gets from them an annuity of £88 3s. 4d. for the remainder of his life, the first payment to be made when he is 61. Reckoning 3 per cent. per annum compound interest, find how long he is expected to live. (R.S.A.)

21. An immediate annuity-certain of £285 8s. per annum is purchased for £3764. Calculate the number of annual payments of the annuity to be made, reckoning compound interest at $3\frac{1}{2}$ per cent. per annum.

22. A man pays an annual premium of £51 4s. in quarterly instalments, beginning on January 1st, 1917, for an annuity of £130 per annum, also to be paid quarterly, the first to become due on January 1st, 1937. Reckoning compound interest at $2\frac{1}{2}$ per cent. per annum, calculate the number of annuity payments to be made if the final premium was paid on October 1st, 1936. Take $\log 1\cdot00625 = 0\cdot0027059$.

23. What sum of money must a man, on his 35th birthday, pay to an insurance company in order to buy an annuity of £200 per annum, the first payment of the annuity to be on his 65th birthday? Assume that the calculation is at the rate of 3 per cent. per

annum compound interest and that the probable number of annual payments of the annuity will be ten. (R.S.A.)

24. A life assurance company agrees to pay £500 to a man on reaching the age of 60, in return for the payment of an annual premium, the first and last to be made on his 22nd and 59th birthdays respectively. Calculate the annual premium, reckoning compound interest at $2\frac{1}{2}$ per cent. per annum.

CHAPTER XVIII

FURTHER MENSURATION

18.1. The Pyramid.

A SOLID standing on a plane rectilineal base and having plane triangular faces which meet in a common vertex is called a **pyramid**. According to the shape of the base, so the pyramid is named. A triangular pyramid is called a **tetrahedron**.

When the sloping faces of a pyramid are all equal isosceles triangles, so that the base is a regular rectilineal figure, the pyramid is said to be **regular**. The line drawn from the vertex perpendicular to the base of such a solid is called its **axis**, and clearly the axis meets the base in the point, which is the centre of the circle circumscribing the regular rectilineal figure, as in Fig. 18.

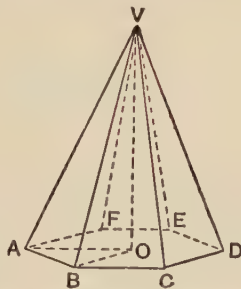


FIG. 18.—Regular hexagonal pyramid.

It may be proved that the volume of any pyramid is equal to one-third of that of a right prism standing on the same base and of the same altitude, i.e.

$$\text{volume of a pyramid} = \frac{1}{3} \times (\text{area of base}) \times \text{height}.$$

Ex. 1. Find the weight, in tons, of a solid pyramid of granite, 6 ft. 6 in. high, standing on a rectangular base measuring 5 ft. 4 in. by 3 ft. 9 in., taking the weight of one cubic foot of granite as 168 lb.

By the above rule, the volume of the pyramid is

$$\frac{1}{3} \times 5\frac{1}{3} \times 3\frac{3}{4} \times 6\frac{1}{2} \text{ cubic feet ;}$$

\therefore the weight $= \frac{1}{3} \times 5\frac{1}{2} \times 3\frac{3}{4} \times 6\frac{1}{2} \times 168$ lb.

$$= \frac{1}{3} \times 5\frac{1}{2} \times 3\frac{3}{4} \times 6\frac{1}{2} \times 168 \div 2240 \text{ tons}$$

$$= \frac{16 \times 15 \times 13 \times 168}{3 \times 3 \times 4 \times 2 \times 2240} \text{ tons} = \frac{13}{4} \text{ tons, on cancelling.}$$

Hence, the required weight $= 3\frac{1}{4}$ tons.

18.2. The Right Circular Cone.

A pyramid whose base is circular is known as a **cone**; when the **axis**, i.e. the line joining the vertex of the cone to the centre of the base, is perpendicular to the base, the solid is called a **right circular cone**.

If r = radius of base and h = vertical height, then, by Section 18.1, the volume of the cone is

$$\frac{1}{3} \times (\text{area of base}) \times \text{height.}$$

But the area of the base = area of circle of radius $r = \pi r^2$.

$$\therefore \text{Volume of Cone} = \frac{1}{3} \pi r^2 h.$$

Usually the diameter is given in most practical cases, and it may therefore be convenient to express the volume in terms of the diameter.

Let the diameter be d , then $r = \frac{1}{2}d$, so that $r^2 = \frac{1}{4}d^2$.

$$\therefore \text{Volume of cone} = \frac{1}{12} \pi d^2 h.$$

It must be carefully observed that, in calculating the volume of a cone, r and h , or d and h must be expressed in the same units.

Ex. 2. *An iron hopper in the form of an inverted hollow cone has an internal base diameter of three feet and an internal vertical height of 4.9 feet. Taking 277.2 cubic inches as the approximate volume of a gallon, and $\pi = 3\frac{1}{7}$, calculate the capacity of the hopper in gallons.*

Here $d = 3$ and $h = 4.9$;

$$\therefore \text{capacity of vessel} = \frac{1}{12} \times \frac{22}{7} \times \frac{9}{1} \times \frac{49}{10} \text{ cubic feet.}$$

$$\begin{aligned}
 &= \frac{1}{12} \times \frac{22}{7} \times \frac{9}{1} \times \frac{49}{10} \times \frac{17280}{2772} \text{ gallons} \\
 &= \frac{22 \times 9 \times 49 \times 17280}{12 \times 7 \times 10 \times 2772} \text{ gallons} \\
 &= 72 \text{ gallons.}
 \end{aligned}$$

18.3. The Frustum of a Pyramid or Cone.

When a pyramid or cone is cut by a plane parallel to its base and the upper part containing the vertex is removed, the remaining

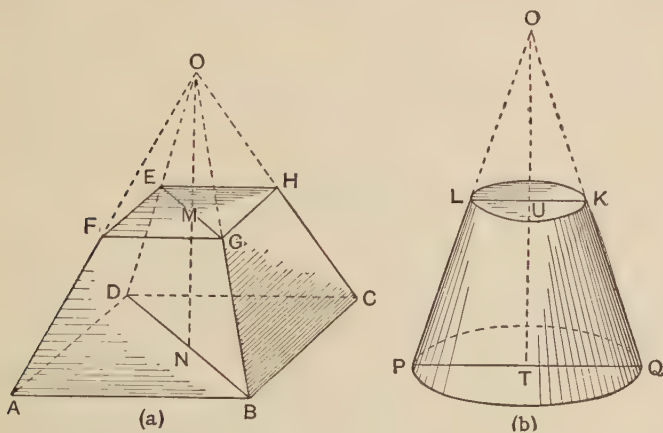


FIG. 19.

solid is called a **frustum** of the pyramid or cone. Thus in Fig. 19, $ABC \dots GH$ is a frustum of a pyramid (a), and $PQKL$ is a frustum of a cone (b).

If the area of the face $EFGH$ (Fig. 19a) be denoted by a , the area of the base $ABCD$ by A and the perpendicular distance NM between these parallel faces by h , then the volume of the frustum $ABC \dots GH$ is given by the expression $\frac{1}{3}h(A + \sqrt{Aa} + a)$; thus the volume of a frustum of a pyramid bounded by two parallel faces of areas A , a respectively and distant h apart is $\frac{1}{3}h(A + \sqrt{Aa} + a)$.

In the case of a frustum of a right circular cone, if R , r be the radii of the parallel faces, then $A = \pi R^2$ and $a = \pi r^2$, so that the expression $\frac{1}{3}h(A + \sqrt{Aa} + a)$ becomes

$$\frac{1}{3}h(\pi R^2 + \sqrt{\pi^2 R^2 r^2} + \pi r^2) = \frac{1}{3}\pi h(R^2 + Rr + r^2);$$

hence, the volume of a frustum of a right circular cone bounded by two parallel circular faces of radii R , r respectively and distant h apart is

$$\frac{1}{3}\pi h(R^2 + Rr + r^2).$$

A simple proof of this formula is quite easy by algebra.

In Fig. 19b, let $LU = r$, $PT = R$, $UO = x$, $TO = H$ and $TU = h$.

Then, it is evident that,

$$\frac{UO}{TO} = \frac{LU}{PT} \quad \text{or} \quad \frac{x}{H} = \frac{r}{R};$$

$$\therefore x = \frac{rH}{R}.$$

Now volume of frustum $PQKL = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 x$

$$= \frac{1}{3}\pi \left(R^2 H - \frac{r^2 H}{R} \right), \text{ substituting for } x,$$

$$= \frac{1}{3}\pi \frac{H}{R} (R^3 - r^3)$$

$$= \frac{1}{3}\pi \frac{H}{R} (R - r)(R^2 + Rr + r^2).$$

But $h = TU = TO - UO = H - x = H - \frac{rH}{R} = \frac{H}{R}(R - r).$

$$\therefore \text{Volume of frustum} = \frac{1}{3}\pi h(R^2 + Rr + r^2).$$

Ex. 3. *A reservoir, in the form of a hollow conical frustum, has the following internal measurements :*

Depth = 50.6 ft., diameters of circular base and top = 200 ft. and 360 ft. respectively.

Calculate the capacity of the reservoir in gallons, to the nearest million, having given that one cubic foot is equivalent approximately to 6.25 gallons.

From the formula just proved,

$$h=50.6, \quad R=180 \quad \text{and} \quad r=100,$$

these measurements being in feet.

Hence, the capacity of the reservoir, in cubic feet,

$$\begin{aligned} &= \frac{1}{3}\pi \times 50.6 \{ (180)^2 + (180 \times 100) + (100)^2 \} \\ &= \frac{1}{3}\pi \times 50.6 (32400 + 18000 + 10000) \\ &= \frac{1}{3}\pi \times 50.6 \times 60400. \end{aligned}$$

\therefore Capacity in gallons

$$\begin{aligned} &= \frac{1}{3}\pi \times 50.6 \times 60400 \times 6.25 \\ &= 19,992,903 \text{ with } \pi=3.14, \end{aligned}$$

or $20,003,091$ with $\pi=3.1416$.

Hence, to the nearest million gallons, the capacity is

20,000,000 gallons.

18.4. Circular Sectors.

The figure bounded by two radii of a circle and the arc between them is called a *sector* of the circle; thus POQ (Fig. 20) is a circular sector. The angle POQ subtended at the centre by the arc is called the *angle of the sector*.

Now it should be clear that the length of the arc and the area of any circular sector are each proportional to the angle of the sector.

Let r = radius of circle, s = length of arc PQ and θ = angle POQ of the sector; then, since for a complete circle, arc = circumference = $2\pi r$ when the angle at the centre is four right angles or 360° ,

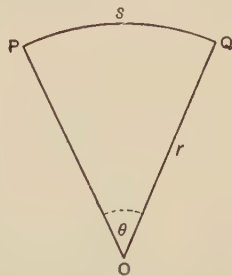


FIG. 20.

$$\frac{s}{2\pi r} = \frac{\theta}{360}, \quad \text{or} \quad s = \frac{\pi r \theta}{180} \dots\dots\dots (i)$$

Again, the area of the whole circle $= \pi r^2$, so that, if A = area of sector,

$$\frac{A}{\pi r^2} = \frac{\theta}{360}, \text{ or } A = \frac{\pi r^2 \theta}{360} \dots\dots\dots(ii)$$

Note that, since $\frac{\pi r \theta}{360} = \frac{s}{2}$ and $A = \frac{\pi r \theta}{360} \cdot r$;

$$\therefore A = \frac{1}{2}sr. \dots\dots\dots(iii)$$

This formula is readily found directly by applying the rule for the area of a triangle to the sector POQ ; for the curved base PQ has a length s and the altitude = radius $= r$. Hence the area $A = \frac{1}{2}sr$.

18.5. Area of a Circular Segment.

The figure bounded by the chord of a circle and the arc it cuts off is called a *segment* of a circle. Thus PRQ , PTQ (Fig. 21) are the

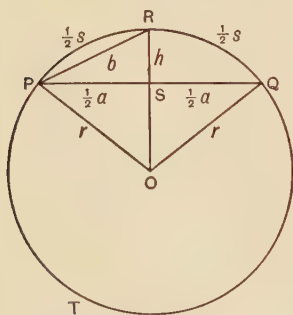


FIG. 21.

two segments into which the chord PQ divides the circle. The arc PRQ , which is the smaller, is called the *minor arc*, whilst the larger arc PTQ is called the *major arc*. The segments PRQ , PTQ are similarly distinguished as the *minor* and *major* segments respectively.

Let R (Fig. 21) be the mid-point of the minor arc PQ and suppose the radius OR cuts PQ in S ; then the area of the segment PRQ = area of sector POQ - area of triangle $POQ = \frac{1}{2}sr - \frac{1}{2} \cdot PQ \cdot OS$. This area can be readily calculated when the lengths of PQ , OS are either given or can be conveniently measured. The general method of calculating the area of the segment involves trigonometry, but this will not be considered here.*

The chord PQ is called the *chord of the arc*, PR the *chord of half the arc*, and SR the *height of the arc*.

* See pp. 393-8 of the author's *Mathematics for Technical Students* (Macmillan).

Now if the lengths of PQ , PR and SR be denoted by a , b and h respectively, approximate formulae have been found for s and A , the area of the segment, in terms of a , b and h , which are the most conveniently measurable lengths. These formulae are :

$$s = \frac{8b - a}{3}, \dots\dots\dots(\text{iv})$$

$$A = \frac{h(4a^2 + 3h^2)}{6a}. \dots\dots\dots(\text{v})$$

It must be remembered, however, that these formulae are not exact but, as long as θ , the angle of the segment, is acute, they give quite reliable results. Indeed (v) may be used for angles less than 180° .

Ex. 4. *The section PQR of a tunnel is shewn in Fig. 22. If $PO = 3\cdot9$ cm., O being the centre of the circular arc, and $SR = 5\cdot9$ cm., calculate the area of the cross-section in square feet, assuming that the scale is 1 cm. to 2 feet and, by measurement from the scale drawing, $\angle POR = 120\cdot85^\circ$. Hence, find what weight, in tons, of earth must be removed to construct a tunnel 378 yards long having this section, taking 162 lb. as the weight of one cubic foot of earth.*

Since the radius $OP = 3\cdot9$ cm., it represents an actual length of $(3\cdot9 \times 2)$ ft. = $7\cdot8$ ft., and similarly for the other measurements.

Also, since $\angle POR = 120\cdot85^\circ$, the angle of the sector $POQR = 120\cdot85^\circ \times 2 = 241\cdot7^\circ$.

\therefore by (ii), area of sector $POQR$

$$= \frac{\pi \times (7\cdot8)^2 \times 241\cdot7}{360} \text{ sq. ft.} = 128\cdot3 \text{ sq. ft.}$$

To this must be added the area of the triangle POQ , and this involves finding the length of PQ .

From the right-angled triangle PSO ,

$$\begin{aligned} PS^2 &= OP^2 - OS^2 = OP^2 - (SR - OR)^2 \\ &= (7\cdot8)^2 - (4)^2 = 60\cdot84 - 16 = 44\cdot84. \end{aligned}$$

$$\therefore PS = \sqrt{44\cdot84} = 6\cdot7, \text{ correct to one place.}$$

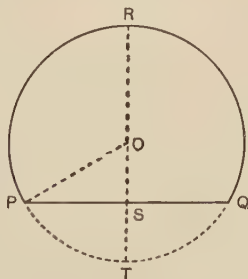


FIG. 22.

Hence, area of triangle $POQ = 6.7 \times 4$ sq. ft. $= 26.8$ sq. ft.

\therefore Total area of section $= (128.3 + 26.8)$ sq. ft. $= 155.1$ sq. ft.

To check this result, suppose the circle to be completed and RS produced to meet the circumference at T ; then, since $\angle POQ$ is less than 180° , for it is $360^\circ - 241.7^\circ = 118.3^\circ$, the approximate formula (v) may be applied to find the area of the minor segment PTQ . For this, $a = PQ = 13.4$ ft. and

$$h = TS = TR - SR = (15.6 - 11.8) \text{ ft.} = 3.8 \text{ ft.}$$

from (v) :

$$\text{area} = \frac{3.8 \{4 \times (13.4)^2 + 3 \times (3.8)^2\}}{6 \times 13.4} \text{ sq. ft.}$$

$$= 35.99 \text{ sq. ft.} = 36 \text{ sq. ft. approximately.}$$

Hence, the area of the major segment PQR

$$= \pi \times (7.8)^2 - 36 \text{ sq. ft.} = (191.1 - 36) \text{ sq. ft.}$$

$$= 155.1 \text{ sq. ft. as previously found.}$$

Finally, the required weight of earth to be removed for a tunnel 378 yd. or 1134 ft. long is

$$\frac{155.1 \times 1134 \times 162}{2240} \text{ tons} = 12,720 \text{ tons.}$$

18-6. The Curved Surface of a Right Circular Cone.

Many solids, like pyramids, have rectilinear faces so that the areas of these faces may readily be determined by methods already considered in Chapter X. In a few cases, however, the faces are curved, as for instance, that of a cone. The area of such a curved surface is known as the **lateral area**, and this is usually quite easy to calculate when the fundamental dimensions of the solid are given.

Let VPQ (Fig. 23a) represent a right circular cone whose sloping length, PV or QV —called the **slant height**—is l , and base radius $OQ = r$.

Now if a thin piece of paper be cut so that when wrapped round the curved surface it completely covers that surface without over-

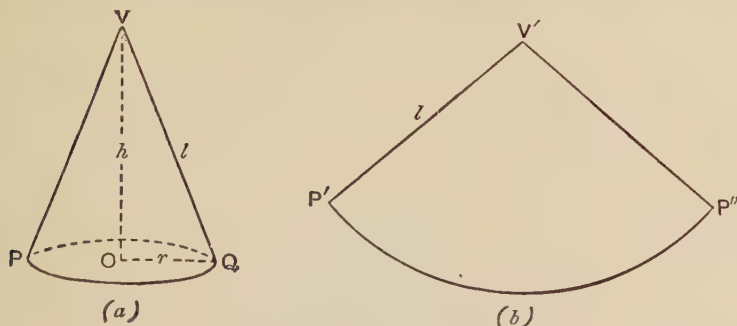


FIG. 23.—Curved surface of a right circular cone.

lapping, then on flattening out the paper it takes the shape $V'P'P''$ (Fig. 23*b*), where $V'P' = V'P'' = VP = l$, and the arc $P'P'' =$ circumference of the circular base PQ .

$V'P'P''$ is, therefore, the sector of a circle, and by Section 18.4 its area is $\frac{1}{2} \cdot V'P' \times \text{arc } P'P''$, i.e. $\frac{1}{2}(\text{slant height}) \times (\text{circumference of base})$. Hence, the lateral surface of a right circular cone is measured by the product,

$$\frac{1}{2}(\text{slant height}) \times (\text{circumference of base}).$$

The slant height

$$l = VQ = \sqrt{QO^2 + OV^2} = \sqrt{r^2 + h^2},$$

and circumference of base $= 2\pi r$;

\therefore lateral surface of a right circular cone of height h , slant height l and base radius r is $\pi r l = \pi r \sqrt{r^2 + h^2}$.

18.7. Curved Surface of a Conical Frustum.

By algebra, the formula for the curved surface of a conical frustum may readily be found.

Referring to Fig. 19*b*, page 269, let $PL = l$, $LO = k$, $PT = R$ and $LU = r$; then

$$\frac{LO}{LU} = \frac{PO}{PT} \quad \text{or} \quad \frac{k}{r} = \frac{l+k}{R}.$$

$$\therefore Rk = rl + rk, \quad \text{or} \quad k(R - r) = rl.$$

Now the curved surface of the frustum $PQKL$

$$\begin{aligned} &= \pi R \cdot PO - \pi r \cdot LO = \pi R(l+k) - \pi rk \\ &= \pi Rl + \pi(R-r)k = \pi Rl + \pi rl, \text{ from the above relation,} \\ &= \pi(R+r)l. \end{aligned}$$

Hence, the lateral area of a frustum of a right circular cone is measured by the product

$$\pi \times (\text{sum of radii of parallel faces}) \times (\text{slant height}).$$

Ex. 5. *A lamp shade is to be made in the form of a conical frustum 4 inches deep, having upper and lower diameters of 2 in. and 17 in. respectively. Allowing 5 per cent. of the net area for overlap and waste, calculate, to the nearest penny, the cost of the material at 1s. 9d. per square yard.*

Let $ABCD$ (Fig. 24) be the vertical section of the shade through

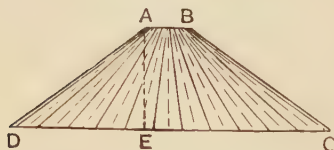


FIG. 24.

its axis; then $AB=2$ in., $CD=17$ in., and the perpendicular distance AE between AB and $CD=4$ in.

Now the lateral area

$$= \pi \times (1+8.5) \times DA \text{ sq. in.}$$

But, from the right-angled triangle DEA ,

$$DA^2 = DE^2 + EA^2 = (7.5)^2 + (4)^2 = 56.25 + 16 = 72.25;$$

$$\therefore DA = \sqrt{72.25} = 8.5.$$

Hence, the net lateral area $= \pi \times 9.5 \times 8.5$ sq. in.

Allowing 5% increase on this for overlap, etc.,

$$\text{area of material required} = \pi \times 9.5 \times 8.5 \times 1.05 \text{ sq. in.}$$

$$= \frac{\pi \times 9.5 \times 8.5 \times 1.05}{144} \text{ sq. ft.}$$

$$\therefore \text{Cost of material} = \frac{\pi \times 9\cdot5 \times 8\cdot5 \times 1\cdot05 \times 1\cdot75}{144} \text{ shillings}$$

$$= 3\cdot236 \text{ shillings}$$

$$= 3\text{s. } 3\text{d. to the nearest penny.}$$

18·8. The Sphere.

A solid whose surface is such that every point on it is equidistant from a fixed point within it is called a **sphere** (Fig. 25). The fixed

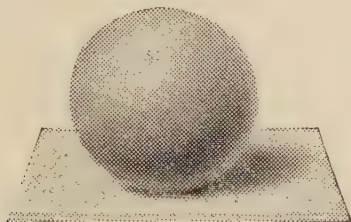


FIG. 25.

point is called the **centre**, and the constant distance of every point on the surface from the centre is called the **radius** of the sphere. Any straight line terminated by the surface and passing through the centre is called a **diameter**. If a sphere is cut by a plane, the surface of the cut portion is called a **plane section** of the sphere.

If r = radius of sphere, then the volume is $\frac{4}{3}\pi r^3$, or if d = diameter, since $r = \frac{1}{2}d$, the volume is $\frac{1}{6}\pi d^3$.

Further, taking $\pi = 3\cdot1416$, $\frac{1}{6}\pi = 0\cdot5236$, so that $\frac{1}{6}\pi d^3 = 0\cdot5236d^3$.

Hence, the volume of a sphere of radius r or diameter d is

$$\frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3 = 0\cdot5236d^3.$$

In the case of a *hollow* sphere, whose internal and external radii are r , R respectively, or diameters d , D , the volume is

$$\frac{4}{3}\pi(R^3 - r^3) = \frac{1}{6}\pi(D^3 - d^3) = 0\cdot5236(D^3 - d^3).$$

Note that the thickness of the material is $R - r = \frac{1}{2}(D - d)$.

Although the proof of the formula for the volume of a sphere is beyond the scope of this book, it may be observed that, if a cylinder

be made just to contain the sphere, i.e. the circumscribing cylinder, then the volume of the sphere is two-thirds that of this cylinder; for evidently the radius of the circular section of the cylinder will be r and its height $2r$. Hence, volume of cylinder $= \pi r^2 \times (2r) = 2\pi r^3$; therefore volume of inscribed sphere $= \frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3$.

Ex. 6. *A sphere of brass whose diameter is 8.3 inches weighs 92.73 lb. Calculate the weight of brass per cubic foot.*

Here $d = 8.3$ in., so that the volume of the sphere

$$= 0.5236 \times (8.3)^3 \text{ cu. in.} = \frac{0.5236 \times (8.3)^3}{1728} \text{ cu. ft.}$$

Hence, since the weight of the sphere is 92.73 lb., the weight of brass per cubic foot is

$$\frac{92.73 \times 1728}{0.5236 \times (8.3)^3} = 535, \text{ by logarithms.}$$

\therefore Weight of brass per cubic foot = 535 lb.

Ex. 7. *Calculate the diameter of a hemispherical basin which has to be made to hold 7.3 gallons, taking 6.25 gallons to a cubic foot. Give the result in feet correct to three significant figures.*

Let the required diameter be d feet, then the volume of the bowl is $\frac{1}{2} \times 0.5236d^3$ cu. ft. $= 0.2618d^3$ cu. ft.

$$\therefore \text{Capacity} = 0.2618d^3 \times 6.25 \text{ gallons,}$$

$$\text{so that } 0.2618d^3 \times 6.25 = 7.3.$$

$$\text{Hence, } d^3 = \frac{7.3}{0.2618 \times 6.25}.$$

Taking logarithms :

$$\begin{array}{rcl} & \log 0.2618 = \bar{1}.4179 & \\ & \log 6.25 = 0.7959 & \\ \log 7.3 = 0.8633 & & \hline & & 0.2138 \\ 0.2138 & \leftarrow & \\ 0.6495 & = \log d^3 \text{ or } 3 \log d. & \end{array}$$

$$\therefore \log d = 0.6495 \div 3 = 0.2165 = \log 1.646.$$

Hence, the required diameter, correct to three significant figures is 1.65 ft.

18.9. Portions of a Sphere.

The portion of a sphere cut off between two parallel planes is called a **spherical frustum**, and the lateral area or curved surface of such a frustum is known as a **spherical zone**. The constant distance between the parallel planes is called the **thickness** of the frustum.

The following formulae, which are very important, are given for reference, without proof.

(i) The volume of a spherical frustum of thickness h and whose plane parallel faces have radii a , b respectively is

$$\frac{1}{6}\pi h(3a^2 + 3b^2 + h^2).$$

Note that, when the frustum becomes a hemisphere of radius r , $a = r$, $b = 0$ and $h = r$, so that the volume of a hemisphere

$$= \frac{1}{6}\pi r(3r^2 + r^2) = \frac{2}{3}\pi r^3.$$

(ii) The area of a spherical zone of thickness h is $2\pi rh$, where r is the radius of the sphere.

When the sphere is cut into two portions by a single plane, either portion is called a **spherical segment**, and its curved surface is known usually as a **spherical cap**.

For a hemisphere, $h = r$; thus the area of the curved surface of a hemispherical cap is $2\pi r^2$, hence :

(iii) The surface area of a sphere of radius r or diameter d is

$$4\pi r^2 = \pi d^2.$$

(iv) The volume of a spherical segment whose greatest thickness or height is h and whose circular base has a radius a is

$$\frac{1}{6}\pi h(3a^2 + h^2).$$

This expression may be put into terms of the radius r of the sphere. Regarding Fig. 21, page 272, as a plane section of a sphere through its centre O ; $OP = OR = OQ = r$, $PS = SQ = a$ and $SR = h$; hence,

$$r^2 = OP^2 = OS^2 + PS^2 = (r - h)^2 + a^2 = r^2 - 2rh + h^2 + a^2,$$

so that

$$a^2 = 2hr - h^2 ;$$

$$\therefore \frac{1}{6}\pi h(3a^2 + h^2) = \frac{1}{6}\pi h(6hr - 3h^2 + h^2) = \frac{1}{3}\pi h^2(3r - h).$$

Hence,

$$\text{volume of spherical segment} = \frac{1}{6}\pi h(3a^2 + h^2) = \frac{1}{3}\pi h^2(3r - h),$$

where h = height, and a = radius of base of segment, and r = radius of sphere of which the segment forms part.

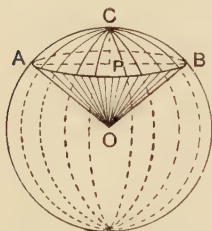


FIG. 26.—Spherical sector.

That portion of a sphere made up by a spherical segment, such as ABC (Fig. 26) and the right circular cone ABO is known as a spherical sector.

(v) The volume of a spherical sector is

$$\frac{1}{3}(\text{area of cap}) \times (\text{radius of sphere}).$$

Now, from (ii), p. 279, the area of a spherical cap of height h is $2\pi rh$;

$$\therefore \text{volume of spherical sector} = \frac{2}{3}\pi r^2 h.$$

Note that in Fig. 26, $PC = h$.

Ex. 8. Find the cost of plating a sphere of diameter 17·6 inches at 8s. 3d. per square foot.

By (iii) of Section 18·9, the area of the surface of the sphere is

$$\pi \times (17\cdot6)^2 \text{ sq. in.} = \frac{\pi \times (17\cdot6)^2}{144} \text{ sq. ft.}$$

$$\therefore \text{Cost of plating} = \frac{\pi \times (17\cdot6)^2 \times 8\cdot25}{144} \text{ shillings}$$

$$= 55\cdot74 \text{ shillings} = \text{£}2 \text{ 15s. 9d.}$$

18·10. Solids of Revolution.

Symmetrical solids with curved surfaces may be considered to be generated by the complete revolution, i.e. through 360° , of a plane figure about a convenient axis in its plane. The following are a few simple cases.

(i) A rectangle $ABCD$ revolved about the side AB generates a right circular cylinder whose height is AB and radius of section AD or BC .

(ii) A triangle PQR having a right angle at Q revolved about PQ as axis generates a right circular cone of height PQ and base radius QR . When revolved about the hypotenuse PR it generates a double cone whose common base radius is equal in length to that of the perpendicular drawn from Q to PR .

(iii) A semicircle revolved about its diameter generates a sphere.

Compound solids may likewise be generated by a suitable combination of plane figures.

Solids formed in this way are known as **Solids of Revolution**.

Ex. 8. The trapezium $ABCD$ (Fig. 27) is revolved about the line XY which is parallel to AB . Calculate, in cubic feet, the volume of the solid thus generated, if $a = XA = YB = 2.6$ in., $R = AD = 7.3$ in., $r = BC = 2.8$ in., and $h = AB = 17.6$ in.

The solid generated is the frustum of a right circular cone whose plane circular faces have radii $(a + R)$, $(a + r)$ respectively and distant h apart, with a cylindrical hole of radius a cut axially through it.

Hence, by Section 18.3, the required volume is

$$\frac{1}{3}\pi h\{(a + R)^2 + (a + R)(a + r) + (a + r)^2\} - \pi a^2 h \\ = \frac{1}{3}\pi h\{(a + R)^2 + (a + R)(a + r) + (a + r)^2 - 3a^2\}.$$

Now,

$$a = 2.6 \text{ in.}, h = 17.6 \text{ in.},$$

$$a + r = (2.6 + 2.8) \text{ in.} = 5.4 \text{ in.},$$

$$a + R = (2.6 + 7.3) \text{ in.} = 9.9 \text{ in.}$$

\therefore Volume

$$= \frac{1}{3}\pi \times 17.6 \times \{(9.9)^2 + (9.9 \times 5.4) + (5.4)^2 - 3 \times (2.6)^2\} \text{ cu. in.}$$

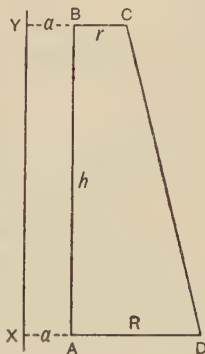


FIG. 27.

$$\begin{aligned}
 &= \frac{1}{3}\pi \times 17.6 \times (98.01 + 53.46 + 29.16 - 20.28) \text{ cu. in.} \\
 &= \frac{1}{3}\pi \times 17.6 \times 160.35 \text{ cu. in.} = \pi \times 17.6 \times 53.45 \text{ cu. in.} \\
 &= \frac{\pi \times 17.6 \times 53.45}{1728} \text{ cu. ft.} = 1.71 \text{ cu. ft.}
 \end{aligned}$$

Ex. 9. In Fig. 28, $ABDE$ is a rectangle and CD a circular arc. When revolved about AC the figure generates a hollow cylindrical tank with a spherical roof. If $a = AE = 1.5$ ft., $k = AB = 3.4$ ft. and $h = BC = 1.2$ ft., calculate the capacity of the tank in gallons, taking 6.25 gallons to a cubic foot. Deduce the capacity when $BC = BD$.

The rectangle $ABDE$ will generate a right circular cylinder whose volume $= \pi a^2 k$, and the triangle BCD with the circular arc CD as a side will generate a spherical segment of height h and base radius a . Hence, by (iv) of Section 18.9, its volume $= \frac{1}{6}\pi h (3a^2 + h^2)$.

\therefore Total volume generated

$$= \pi a^2 k + \frac{1}{6}\pi h (3a^2 + h^2).$$

Now, $a = 1.5$ ft., $h = 1.2$ ft. and $k = 3.4$ ft.

\therefore Volume in cubic feet

$$\begin{aligned}
 &= \pi \times (1.5)^2 \times 3.4 + \frac{1}{6}\pi \times 1.2 \times \{3(1.5)^2 + (1.2)^2\} \\
 &= (\pi \times 7.65 + \pi \times 0.2 \times 8.19) \\
 &= \pi (7.65 + 1.638) = \pi \times 9.288.
 \end{aligned}$$

\therefore Capacity $= \pi \times 9.288 \times 6.25$ gallons

$$= 182.4 \text{ gallons.}$$

When $BC = BD$, or $h = a$, DBC is a quadrant of a circle, which, on revolution about BC , generates a hemisphere.

Hence, total volume generated, in cubic feet,

$$= \pi a^2 k + \frac{2}{3}\pi a^3 = \pi a^2 (k + \frac{2}{3}a) = \pi \times (1.5)^2 \times 4.4.$$

\therefore Capacity, in gallons, $= \pi \times (1.5)^2 \times 4.4 \times 6.25$

$$= 194.4.$$

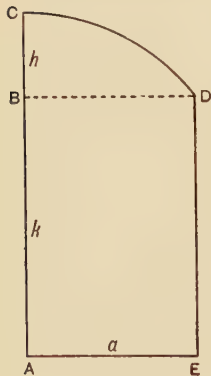


FIG. 28.

EXERCISES 18

In the following exercises, Nos. 1-16, no logarithms are to be used and the value of π is to be taken as $3\frac{1}{7}$.

1. By what percentage must the height of a right circular cone be diminished when the diameter of its base is increased by 15 per cent. so that the volume may be unchanged?

If the base diameter were decreased by 15 per cent. and the height increased by 15 per cent., by what percentage would the volume be diminished?

2. $PQRS$ is a square whose side is 2 ft. 4 in. long. With centre P and radius PQ an arc SEQ is described and with centre R and radius RS an arc SFQ is described, both arcs being inside the square. Find the areas enclosed by (i) the sides PQ , PS and the arc SFQ , and (ii) the two arcs SEQ , SFQ . (C.P.)

3. A cylindrical tin 5 inches high holds a quart. Find the area of the curved surface to the nearest square inch, assuming that 1 quart = 69.32 cubic inches. (L.Ch.C.)

4. A thin piece of paper is cut so that it just covers the curved surface of a right circular cone. When opened out flat the paper forms exactly four-fifths of the circle whose radius is 17.5 cm. Calculate the volume of the cone in cubic centimetres. (C.P.)

5. The area of the curved surface of a cylinder is twice the area of the base. If the curved surface and the base have together an area of 14.4 square feet, find the volume in cubic feet correct to one decimal place. (L.Ch.C.)

6. A rectangular block of metal 27.5 in. by 10.5 in. by 6 in. is melted down and then cast into a number of spherical balls, each 1.5 inches in diameter. Assuming there is no loss in volume during the process, calculate the number of spherical balls cast. (C.P.)

7. A solid metal sphere has a diameter of 3 ft. 9 in. Calculate (i) the weight of the sphere in tons, etc., if 1 cubic foot of the metal weighs 448 lb.; (ii) the cost of gilding its surface at 4s. 8d. per square foot. (C.P.)

8. A solid sphere of diameter 10.5 cm. and a right circular cylinder 15.75 cm. in length have equal volumes. Find (i) the diameter of the circular base of the cylinder, and (ii) the difference in the total areas of their surfaces. (C.P.)

9. The well of a glass inkpot has the form of a right circular cylinder $1\frac{1}{2}$ inches in diameter and 3 inches deep surmounting a hemisphere of the same diameter. Calculate how many gallons of ink will be required to fill 42 dozen inkpots of this size, taking 277.2 cubic inches to a gallon.

10. A solid is made by joining the flat face of a hemisphere to the circular base of a right circular cone. If the common diameter is 3 feet and the height of the cone is 2 ft. 3 in., calculate the volume of the solid in cubic feet.

11. A rectangular block of wood 12.3 in. by 10.6 in. by 7.4 in. has a hemispherical cavity, of diameter 8.4 in., cut into its upper face. Calculate (i) the volume of the block in cubic inches, and (ii) the area of the curved surface of the cavity in square inches.

12. A double convex lens may be considered to have the form of two equal spherical segments each having a diameter of 7 cm. and a greatest thickness of 7 mm. The lens weighs 68.3 grams; calculate the weight of one cubic centimetre of the glass.

13. ABC is a triangle in which the perpendicular from B to AC meets AC in N . The triangle is revolved about the side CA , thus generating a spindle. Calculate the weight of the spindle in lb., taking the weight of one cubic foot of the material to be 450 lb. if $NB=3.6$ in., $AN=10.5$ in. and $NC=7.7$ in.

14. A grain bin consists of a hollow cylinder, 9 in. deep and 10 in. in diameter, fitted to a conical frustum of the same diameter which tapers downwards to a diameter of 2 in. The frustum is 6 in. deep. Calculate (i) the capacity of the bin in cubic inches, and (ii) the weight of the grain it holds when full, taking 29 cubic inches as the volume occupied by 1 lb.

15. The weight of an Indian club is 3.34 lb. and, to increase this to 4 lb., a cylindrical hole 3.5 in. long is drilled in the thick end and filled with lead. Find the diameter of the hole, taking one cubic inch of wood and of lead to be 0.032 lb. and 0.407 lb. respectively.

16. A number of spherical steel balls, each 2 in. in diameter, are packed in sawdust in a rectangular wooden box whose external dimensions are 13 in. by 10 in. by 9 in. The wood is half an inch thick and the fully packed box weighs 120.9 lb. Given that the weights of a cubic foot of steel, wood and sawdust are 513 lb., 38.4 lb. and 32.4 lb. respectively, find the number of balls in the box.

In exercises 17-32, four-figure logarithms are to be used and $\log \pi$ taken as 0.4971.

17. A cubical box made of wood half-an-inch thick is filled with sand. Sand weighs 3.4 times as much as the same volume of wood, and the box, when full, weighs nine times as much as when empty. Find the length of the external edge of the box. (R.S.A.)

18. A turret is in the form of a right pyramid standing on a square base whose side is six feet long. The length of each sloping edge is 9.25 ft. Find (i) the vertical height of the pyramid above its base, (ii) the area, in square feet, of the sheet zinc required to cover the four sloping faces of the turret, and (iii) the weight of the zinc in lb. if one square foot weighs 1.2 lb. (C.P.)

19. A conical vessel has a depth of 12 in. and a diameter at the top of 5 in. It is already filled with water to a depth of 4 in. Find how much the surface will be raised if another half-pint of water is poured in. Take the volume of one gallon as 277 cubic inches. (R.S.A.)

20. Find, to the nearest tenth of an inch, the diameter of a solid iron sphere weighing 16 lb., given that one cubic foot of iron weighs 483 lb. (R.S.A.)

21. A uniformly thick hollow copper sphere weighs 113.8 lb. and its external diameter is 18 in. Taking the weight of one cubic foot of copper as 558.1 lb., calculate the thickness of the copper wall of the sphere. (C.P.)

22. A solid sphere of glass, 4 in. in diameter, is packed tightly with sawdust into a cubical box 5 in. each way internally and weighing 2 lb. Given that the total weight is 8.28 lb. and that the weight of sawdust is three-eighths that of an equal volume of glass, find the weight of a cubic foot of glass. (R.S.A.)

23. A glass tumbler, when quite full, is to hold half-a-pint of liquid. The tumbler is in the form of a conical frustum, 2.6 in. in diameter at the top and 2 in. in diameter at the bottom. Given that one gallon is equivalent to 277.2 cubic inches, calculate, correct to two places of decimals, the height of the tumbler.

24. The interior of a flower bowl is an exact hemisphere. Empty it weighs 2 lb. 7 oz., and full of water 11 lb. $6\frac{1}{2}$ oz. If one cubic foot of water weighs 62.3 lb., find, to the nearest tenth of an inch, the interior diameter of the top of the bowl. (B.M.I.)

25. An ordinary pail is in the form of a frustum of a right circular cone. Its depth is 10·8 in., and the internal diameters at the top and bottom are 1 ft. and 9 in. respectively. Find the total weight of the pail when quite full of water, taking 4 lb. as its weight when empty and 62·4 lb. as the weight of one cubic foot of water. (C.P.)

26. What must be the diameter of a sphere in order that its surface may be equal to that of a cube of edge ten centimetres?

Express the volume of the cube as a percentage of the volume of the sphere. (R.S.A.)

27. The diameter of a hemispherical bowl is 20 inches, and it is filled with water from a tap in 25 seconds. At what rate, in gallons per minute, is water supplied from the tap? Assume that the volume of a hemisphere of diameter d inches is $0·2618d^3$ cubic inches and that one cubic foot = $6\frac{1}{4}$ gallons. Give the answer correct to one decimal place. (R.S.A.)

28. A bowl has a circular horizontal section with a symmetrically curved side. When filled with water to a depth of y inches, the volume of the water is $\pi(a^2y + 0·2y^3)$ cubic inches, where a is the radius of the base. Find the number of pints of water the bowl will hold when filled to the brim if the diameter of its base is 8 inches and its depth is $7\frac{1}{2}$ inches, taking 34·7 cubic inches to a pint.

29. $LMNO$ is a trapezium in which ON is parallel to LM and each of these sides is perpendicular to LO . The figure revolves about LO as axis; calculate the volume of the solid thus generated, to the nearest cubic inch, if $LM = 2·4$ in., $ON = 0·9$ in. and $LO = 3·5$ in.

30. $PQRST$ is a four-sided figure made up of a rectangle $PQRT$ and a circular arc RS which meets PT produced in S . The whole figure revolves about PS and thus generates a hollow cylindrical tank with a spherical roof. If $PQ = 42$ cm., $PT = 63$ cm. and $TS = 24$ cm., calculate the capacity of the tank in litres correct to four significant figures.

31. A circular arc CB meets two mutually perpendicular straight lines CA , BA at C and B respectively. The figure revolves about AC , thus generating a bowl in the form of a spherical segment. Calculate the capacity of this bowl, in gallons, when $AB = 8·4$ in. and $CA = 7·2$ in., taking 6·25 gallons to a cubic foot.

32. The volume V in cubic feet of a solid is given by the formula

$$V = \frac{1}{8}h(A + 4B + C),$$

where A , B , C are the sectional areas, in square feet, at the top, half-way down and at the bottom respectively and h is the height in feet.

Apply this formula to find the capacity, in gallons, of a water butt having the following dimensions: Height=2 ft. 5 in., diameter at top=diameter at bottom=1 ft. 7 in., diameter midway between top and bottom=1 ft. 10 in. Take 6.25 gallons to a cubic foot.

CHAPTER XIX

GRAPHICAL REPRESENTATION ON SQUARED PAPER

19.1. The Advantage of Squared Paper Representation.

THE relation between a series of values of two correlated numbers is generally not very apparent from the sets of figures themselves, but may usefully be presented by means of a simple diagram drawn on squared paper. An easy example will, perhaps, best illustrate the method.

Ex. 1. *The prices of a few kettles of different capacities is shewn in the following table :*

<i>Capacity in pints</i> -	2	3	5	7
<i>Price in pence</i> -	46	59	85	111

Illustrate the relation between capacity and price on squared paper and read off from the diagram, (i) the cost of a kettle whose capacity is half-a-gallon, and (ii) the capacity of a kettle costing 8s. 2d.

It should first be noted that the prices are not in the same ratio as the capacities, for

$$\frac{2}{46}, \frac{3}{59}, \frac{5}{85}, \frac{7}{111},$$

are not equal fractions, so that the precise relation of price to capacity is not evident at a glance. Hence, the advantage of squared paper representation.

Take a piece of squared paper, preferably ruled in tenths of an inch, and draw on it two axes OX , OY (Fig. 29) perpendicular to each other. Along OX , choose a convenient scale, say $\frac{1}{2}$ inch to represent a pint, and along OY take $\frac{1}{2}$ inch to represent 10 pence. Mark the scale divisions, 0, 1, 2, ... along OX and 0, 10, 20, ...

along OY . Now take the first pair of numbers, viz. 2 (pints), 46 (pence), and where the ruled vertical line from 2 on OX meets the ruled horizontal line from 46 on OY , mark a point A and draw

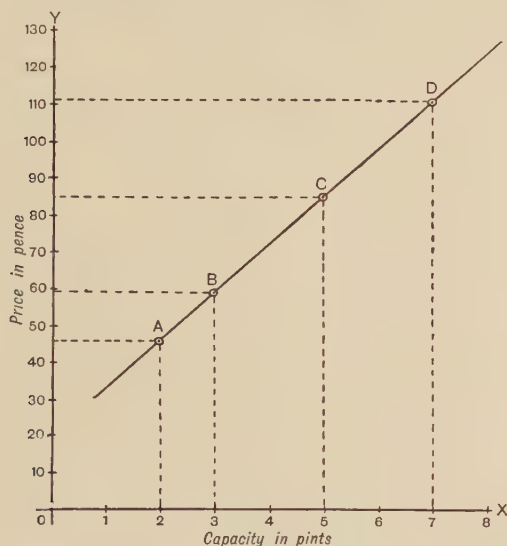


FIG. 29.

a tiny circle round it, as indicated. Proceed similarly with the other pairs, thus obtaining the points B , C , D . Place the straight edge of a ruler along the points, and it will be seen that the four points A , B , C , D lie in a straight line. Draw this line and produce it beyond A and D . The relation between capacity and price is thus revealed by this straight line, and from it the answers to (i) and (ii) may be read off without calculation, as will now be done.

(i) Since $\frac{1}{2}$ gallon = 4 pints, see where the vertical line from 4 on OX meets the line $ABCD$, and from this point trace the horizontal line back to OY ; it meets OY at the point 72; thus the price of a kettle holding half-a-gallon is 72 pence or 6s.

(ii) Since 8s. 2d. = 98 pence, the point on OY representing this price must be found and the method of (i) repeated, with the exception that, this time, it is in the reverse order. The point on OX corresponding to 98 on OY is thus found to be 6; hence, the capacity of a kettle costing 8s. 2d. is 6 pints.

It should be observed that, by producing the graph beyond D , the prices of kettles of greater capacity than 7 pints may be determined and vice versa. Thus the price of a kettle holding a gallon or 8 pints is 124 pence or 10s. 4d.

19-2. The Plotting and Reading of Graphs.

From Ex. 1 it will be clear that each pair of correlated numbers may be represented on squared paper by a point, and the marking of such a point is sometimes called **plotting**. If a point P represents the numbers a , b , it is often convenient to refer to the point as $P(a, b)$, the number a being on the horizontal scale and the number b on the vertical scale.

The line drawn through a series of plotted points is called a **graph**. This need not necessarily be a straight line, as will be seen later.

The process of reading off values from a graph, as in (i) and (ii) of Ex. 1, is known as **Interpolation**.

In plotting a graph from a table of values, there are several important principles which should always be observed, for it must be remembered that the main object of representing a set of corresponding values graphically is to exhibit as much useful information as possible in a form easily and quickly understood. The following rules should therefore be applied in every case.

- (i) The axes should be drawn in thick lines with ink.
- (ii) Write along each axis what it is intended to represent.
- (iii) Choose the scales so that the graph fills the sheet and that the decimal system may be used accurately, i.e. the length of each unit should be a convenient multiple of 5 times the length of a side of the smallest square on the paper.

- (iv) Graduate each axis. Merely writing a scale on the paper does not permit of rapid reading.
- (v) Mark every point in pencil at first and draw a tiny circle round it so that its position may be seen when the graph is drawn.
- (vi) Sketch the curve in pencil through the points plotted and, before inking in, make sure it is a smooth curve, i.e. one having no abrupt changes in direction or sharp angular turns. If the graph represents statistical data, consecutive points are joined by straight lines.

There are other details which only experience in plotting graphs will reveal.

19.3. The Gradient at any Point on a Graph.

In dealing with a set of statistics, it is frequently desirable to examine the rate of increase or decrease at given points. This may be done quite easily from the graph.

Ex. 2. *By the use of squared paper, find which is the greatest and which is the least of the following fractions :*

$$\frac{19}{23}, \frac{23}{29}, \frac{27}{37}, \frac{37}{43}.$$

Verify the results by expressing each fraction in decimal form.

Prepare a sheet of squared paper by drawing the axes OX , OY (Fig. 30) and marking convenient scales along them. Let OX represent the denominators and OY the numerators; then plot the four points $A(19, 23)$, $B(23, 29)$, $C(27, 37)$, $D(37, 43)$. Join AO , BO , CO , DO . Now, for the line OA , the angle aOA is called the slope, and the ratio aA/Oa its gradient, and similarly for the other lines.

Suppose a number of points P, Q, R, S, \dots were plotted representing fractions having the same number as denominator, e.g. $P(11/23)$, $Q(13/23)$, $R(16/23)$, $S(19/23)$, \dots , and each of these points were joined to O , then

$$\angle POX < \angle QOX < \angle ROX < \angle SOX;$$

so that the greater the numerator the greater the slope, and, therefore, the greater the gradient, since each fraction is a measure of the gradient.

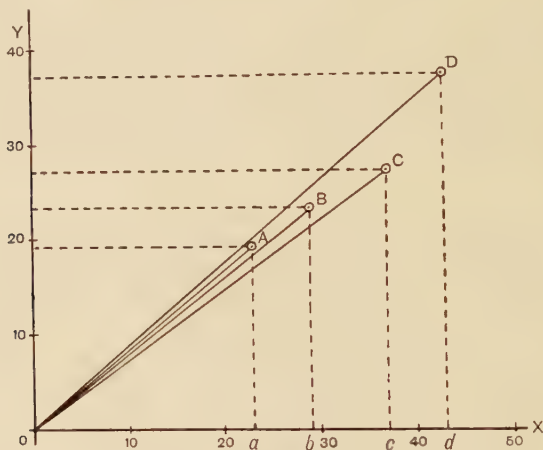


FIG. 30.

Applying this simple fact to the lines OA , OB , OC , OD of Fig. 30, the slopes in descending order of magnitude are $\angle dOD$, $\angle aOA$, $\angle bOB$, $\angle cOC$; therefore the gradients, in descending order of magnitude, are dD/Od , aA/Oa , bB/Ob , cC/Oc , i.e. $37/43$, $19/23$, $23/29$, $27/37$. Hence, of the four fractions,

$\frac{37}{43}$ is the greatest and $\frac{27}{37}$ is the least.

To verify these results, the decimal forms of the fractions are :

$$\frac{19}{23} = 0.8261 ; \quad \frac{23}{29} = 0.7931 ; \quad \frac{27}{37} = 0.7297 ; \quad \frac{37}{43} = 0.8605,$$

so that the greatest is $\frac{37}{43}$ and the least is $\frac{27}{37}$.

Of the other two, $19/23 > 23/29$, which agrees with Fig. 30.

Ex. 3. *The wheat grown in Great Britain, in millions of tons, is shewn for ten years in the following table :*

Year -	1928	1929	1930	1931	1932	1933	1934	1935	1936	1937
Millions of tons	1.32	1.33	1.13	1.01	1.17	1.67	1.86	1.74	1.47	1.31

Illustrate these statistics graphically.

From the graph, determine (i) the slowest and the most rapid rates of increase in the number of tons of wheat grown between any two consecutive years, and (ii) what weight of wheat would have been grown in 1935 if the rate of increase from 1933 to 1934 had been maintained between 1934 and 1935.

Choosing $\frac{1}{2}$ inch as the unit for a year on the horizontal axis, the scale reading beginning at 1928, and $\frac{1}{2}$ inch for 0.1 of a million tons on the vertical axis, the scale beginning at 1.00 million tons, the points *A, B, C, ... L* may easily be plotted and then joined by straight lines as indicated in Fig. 31.

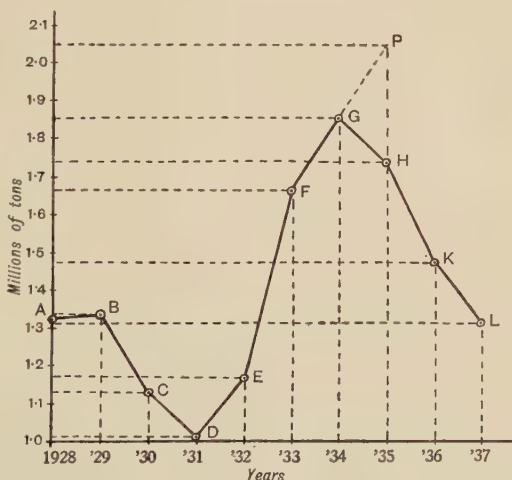


FIG. 31.

The graph thus shews at a glance the variations from year to year, in the tonnage of wheat grown.

(i) From the graph, it is obvious that the line *AB* makes the

smallest angle upwards with the horizontal, so that the slowest rate of increase was from 1928 to 1929.

Similarly, the steepest inclination upwards is shewn by the line EF ; hence, the most rapid rate of increase was from 1932 to 1933.

(ii) If the rate of increase from 1933 to 1934 had been maintained from 1934 to 1935, then the line FG would have been produced to meet the vertical line from 1935. Produce FG , therefore, to meet the vertical from 1935 in P . It will then be seen that the scale reading on the vertical axis corresponding to P is 2·05; hence, the tonnage in 1935 would have been 2·05 million tons.

19·4. Continuous Curves.

In graphs representing statistics, the plotted points are joined by straight lines, as in Fig. 31, because they are **discrete points**, i.e. the values represented by any two points are unrelated. In Ex. 1, however, the values represented by the points are related and the graph, shewn in Fig. 29, though a straight line, is continuous. A graph shewing compound interest or amount is also a continuous curve, though not a straight line, because its growth goes on unceasingly. An example will make this important point clearer.

Ex. 4. *The amounts at various rates of compound interest per annum of £1 for five years are given approximately in the following table :*

Rate per annum -	1%	2%	3%	4%	5%	6%
Amount - -	£1·051	£1·104	£1·159	£1·217	£1·276	£1·338

Represent these values on a graph and, from it, find (i) the rate per cent. per annum when the amount is £1·246, and (ii) the amount when the rate per cent. per annum is $2\frac{1}{2}$.

For graphs of this type, it is advisable to use squared paper ruled in millimetre squares.

Proceeding in the usual manner, the points are easily plotted

when convenient scales have been chosen. The points do not, however, lie in a straight line and, as a consequence, a smooth

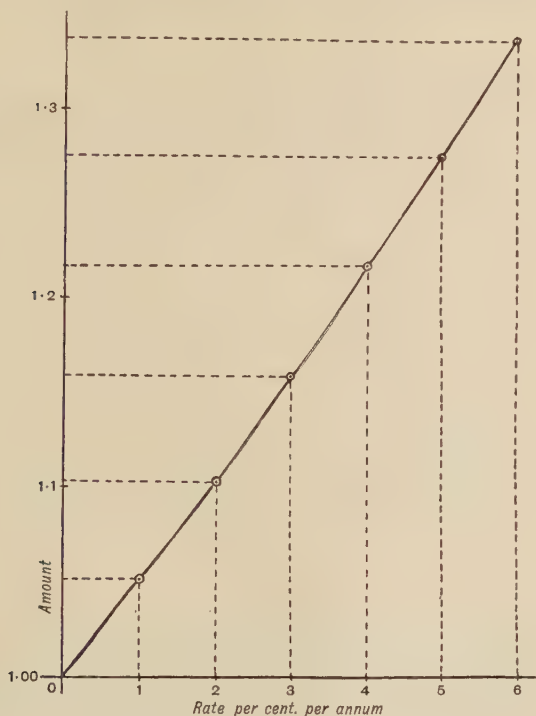


FIG. 32.

continuous curve must be drawn through them, as shewn in Fig. 32.

(i) Looking for 1.246 on the vertical scale and then finding the corresponding point on the horizontal axis, by means of the graph, the value thus determined is $4\frac{1}{2}$.

∴ the rate at which £1 amounts to £1.246 in 5 years is

$4\frac{1}{2}$ per cent. per annum.

(ii) Beginning this time with the point $2\frac{1}{2}$ on the horizontal scale, the corresponding point on the vertical axis is found approximately as 1.131 ; hence, in five years at $2\frac{1}{2}\%$ per annum, £1 amounts to £1.131.

19.5. Roots by the Graphical Method.

The graph (Fig. 32) of the last exercise may also be used for finding the fifth root of a number for, from the compound interest formula, $A = PR^n$; when $P = 1$ and $n = 5$, $A = R^5$ or $R = \sqrt[5]{A}$; hence, from the table of Ex. 4 :

$$1.01 = \sqrt[5]{1.051}, \quad 1.02 = \sqrt[5]{1.104}, \quad 1.03 = \sqrt[5]{1.159}, \dots$$

If therefore, the scale, 1, 2, 3, ... , representing values of r , marked on the horizontal axis were replaced by the corresponding values of R , i.e. 1.01, 1.02, 1.03, ... , the fifth root of any number shewn on the vertical axis could be read off directly from the horizontal scale. For instance, suppose the fifth root of 1.236 were required. Looking for this number on the vertical scale and tracing from the graph the corresponding point on the horizontal axis, the number 1.043 (4.3 on Fig. 32) is thus found. Hence,

$$\sqrt[5]{1.236} = 1.043$$

approximately.

Generally, if a series of corresponding values of x and y , satisfying the equation $y = x^n$, n being an integer, be plotted and a smooth curve drawn through the points, the n th root of any value of y within the scale may be read off as the corresponding value on the axis of x .

19.6. Graphs of Monetary Transactions.

As a final example of the graphical method, the case of an annuity purchased by monthly premiums will be considered.

Ex. 5. *For the purchase of an annuity of £26 per annum at the age of 55, the following monthly premiums are charged according to the age at which the first payment is made :*

Age - - -	25	26	28	31	32	35
Monthly premium	12s.	12s. 7d.	14s.	16s. 7d.	17s. 7d.	21s. 3d.

Represent these data graphically.

From the graph, determine (i) the monthly premium, the first payment of which is to be made at the age of 30, and (ii) the age at which the first monthly premium would be 18s. 8d.

The procedure is exactly as before. In this case, however, it is convenient to express the premiums in pence. From the graph (Fig. 33),

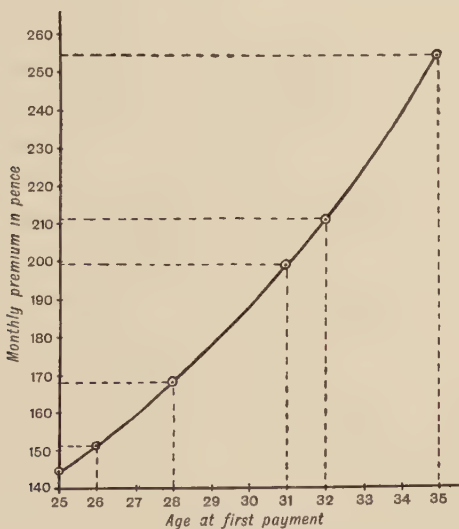


FIG. 33.

- (i) the monthly premium to begin at the age of 30 is 188 pence or 15s. 8d.
- (ii) the monthly premium of 18s. 8d. or 224 pence is chargeable to a person whose age at the time of the first payment is 33.

EXERCISES 19

Each of the following exercises should be worked on a good-sized sheet of accurately ruled squared paper. The rules, stated in Section 19.2, must be carefully observed.

1. If articles are priced at 5s. per 100, construct a simple graph to shew the price of any number of articles up to and including 100.

Read from the graph the price of (i) 29 articles, (ii) 68 articles, (iii) 72 articles. (U.L.C.I.)

2. Given that 1 kilogram = 2.205 lb., shew that, when £1 = 176 francs, then 55 francs per kilogram is approximately equal to 34 pence per lb. Use this statement of comparative prices to draw a graph for converting francs per kilogram into pence per lb. on the rate of exchange given. From the graph read off the British price equivalent to 71.5 francs per kilogram.

3. A load of coal weighing 5 tons 10 cwt. is worth £7 16s. Draw a graph connecting weight of coal, up to six tons, with its value. From the graph read off :

- (i) the value, to the nearest shilling, of 2 tons 8 cwt.,
- (ii) the weight, to the nearest cwt., of coal worth £6 2s.

(U.L.C.I.)

4. By the use of squared paper and the slopes of three straight lines, find which is the greatest of the following ratios :

$$\frac{20}{31}, \frac{29}{43}, \frac{36}{55}.$$

Check the result by expressing the three fractions in decimal forms.

(U.L.C.I.)

5. The cost, C shillings, for n calls on a telephone is given for five quarters in the following table :

n	33	27	51	84	114
C	22.75	22.25	24.25	27	29.5

Shew the relation between C and n on a graph, and from it find :

- (i) the cost of 45 calls, and (ii) the number of calls made when the charge is 25s. 6d.

6. Represent graphically the following extract from the catalogue of an engineering firm, £ C being the price of an engine of H horse-power :

H	4	7	9	16	25	32
C	31	40	46	67	94	115

From the graph, find (i) the price of an engine of 12 horse-power, and (ii) the horse-power of an engine priced at £103.

7. A commodity is sold in tins of varying sizes. The prices of a few sizes are as follows :

Size in ounces	2	4	6	10	14	16
Price - -	11½d.	1s. 1d.	1s. 3½d.	1s. 11½d.	2s. 11½d.	3s. 7d.

By drawing a graph of these data, determine the prices of tins holding 8 oz. and 12 oz. respectively.

8. The table gives the number N of passengers, in thousands, carried each year in British aircraft on journeys between England and abroad :

Year -	1928	1929	1930	1931	1932	1933	1934	1935	1936	1937
N -	24.8	26.2	22.0	21.9	41.6	53.5	58.1	70.0	72.2	78.0

Illustrate these data graphically.

From the graph, determine (i) the slowest rate of increase of the number of passengers between any two years from 1931-1937, and (ii) how many passengers there would have been in 1935 if the rate of increase between 1933 and 1934 had been maintained between 1934 and 1935. (L.Ch.C.)

9. The average price of thirty United States industrial ordinary shares during the months from January to September, 1938, is given below :

Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.
57	54.5	55.2	44.4	49	48.5	60.5	62.5	62

Illustrate these results on a graph.

State from the graph :

- (i) when there was the sharpest fall in share prices,
- (ii) when there was a period of financial depression in the United States,
- (iii) when financial operations were fairly stable. (L.Ch.C.)

10. The following table gives, for each of six years, the value of British imports in millions of pounds :

Year	-	-	-	1931	1932	1933	1934	1935	1936
Total imports in millions	-	-	-	£861	£702	£675	£731	£757	£849

Represent graphically these statistics and, from the graph, find (i) the greatest rate of decline between any two consecutive years, and (ii) what value the imports in 1935 would have reached had the rate of increase between 1933 and 1934 been maintained between 1934 and 1935.

11. The following table gives the amounts, to the nearest 5s., of a number of yearly payments, each of £1, at $2\frac{1}{2}$ per cent. per annum compound interest :

Payments	-	5	8	14	19	26	36	39	45
Amounts	-	£5 5s.	£8 15s.	£16 10s.	£24	£36	£57 5s.	£64 15s.	£81 10s.

Draw a graph shewing the relation between the number of payments and amount. From the graph read off, to the nearest five shillings, the amount of 31 yearly payments. (U.L.C.I.)

12. If £ P invested at simple interest for T years at 5 per cent. per annum amounts to £100, then P and T are connected by the formula

$$P = \frac{2000}{20 + T}.$$

Construct a table giving the values of P when $T=0, 5, 20, 30, 40, 50, 60$. Use the table to draw a graph between T and P . From the graph find, to the nearest year, the value of T when $P=59$. (U.L.C.I.)

13. The approximate compound interest on £100 at $3\frac{1}{4}$ per cent. per annum is given in the following table for a number of years :

No. of years	5	9	14	20	26	32
C.I. - -	£17·3	£33·4	£56·5	£89·6	£129·7	£178·3

No. of years	36	41	44	50
C.I. - -	£216·3	£271·1	£308·5	£394·9

Use these data to construct a graph giving the compound interest up to 50 years. From the graph, determine the compound interest at $3\frac{1}{4}$ per cent. per annum on (i) £100 for 16 years, and (ii) £40 for 28 years.

14. The necessary sinking fund to amount to £100 at $2\frac{1}{2}$ per cent. per annum in a given number of years is shewn in the following table :

No. of years	1	5	10	15	20	25	30	35	40
Sinking Fund	£100	£19·03	£8·93	£5·58	£3·92	£2·93	£2·28	£1·82	£1·49

Draw a graph shewing the relation between the number of years and the sinking fund. Determine from the graph :

- (i) the sinking fund necessary to amount to £100 in 28 years,
- (ii) the number of years in which a sinking fund of £6 12s. will amount to £100. (L.Ch.C.)

15. Find graphically the fifth root of 3·72. Hence, find the rate per cent. per annum compound interest at which £50 will amount to £186 in five years.

16. The following table gives the amount of £200 and the true present value of £700 for periods up to 30 years at 4 per cent. per annum :

No. of years - -	5	10	15	20	25	30
Amount - -	£243	£296	£360	£438	£533	£649
True P.V. - -	£575	£473	£389	£319	£263	£216

Draw a graph connecting amount with time and, with the same axes and scales, draw a second graph connecting true present value with time. From the graphs, read off, to the nearest half-year in each case: (i) the time in which the amount of £200 becomes equal to the present value of £700; (ii) the times in which the difference between the amount and present value becomes £100.

(U.L.C.I.)

17. The table gives the yield per cent. of $3\frac{1}{4}$ per cent. stock at different prices :

Prices of $3\frac{1}{4}\%$ stock

Yield per cent.

	£	s.	d.
130 - - - - -	2	10	0
121 - - - - -	2	13	9
113 - - - - -	2	17	6
104 - - - - -	3	2	6
$97\frac{1}{2}$ - - - - -	3	6	8
91 - - - - -	3	11	5
$84\frac{1}{2}$ - - - - -	3	16	11
78 - - - - -	4	3	4
$71\frac{1}{2}$ - - - - -	4	10	11

Draw a graph to shew the relation between the yield per cent. and the price. From the graph determine :

(i) the price when the yield per cent. is £3 9s.,

(ii) the yield per cent. when the price is £110.

Verify the answer to (ii) by calculation.

(L.Ch.C.)

18. The table gives the annual premiums charged by certain insurance companies to secure the payment of £100 in return for premiums paid for the stated number of years :

Number of years

Annual premium

						£	s.
10	-	-	-	-	-	8	9
14		-	-	-	-	5	14
18	-	-	-	-	-	4	3
22	-	-	-	-	-	3	4
26	-	-	-	-	-	2	10
30	-	-	-	-	-	2	1
34	-	-	-	-	-	1	14
38	-	-	-	-	-	1	8

Draw a graph shewing the relation between the number of years and the annual premium. Determine from the graph :

(i) the annual premium it would be necessary to pay for 19 years to secure £100,

(ii) the number of years in which an annual premium of £22 12s. would secure £1000. (L.Ch.C.)

19. The following statistics are taken from an actuarial table shewing a man's expectation of life at various ages :

Age	-	-	-	25	30	35	40	45	50	55	60	65
Expectation of life in years	-	37·0	33·1	29·2	25·6	22·2	18·9	15·8	12·9	10·3		

Represent this table graphically and read off from the graph :

(i) the expectation of life of a man aged 48,

(ii) the age at which the expectation of life is 14·6 years.

20. The table gives the number of years which a lease has to run and the corresponding number of years' purchase which it is worth, reckoning interest at 6 per cent. per annum :

No. of years	-	5	10	15	20	25	30	35	40
Years' purchase	-	4·21	7·36	9·71	11·47	12·78	13·76	14·50	15·05

Draw a graph shewing the relation between the number of years still to run and the number of years' purchase. Determine from the graph :

- (i) the number of years a lease should still have to run if the number of years' purchase at 6 per cent. is 12·3 ;
 (ii) the value of a lease with 26 years to run if one year's purchase is worth £150, reckoning interest at 6 per cent.
 (L.Ch.C.)

21. The table gives the surrender value of a £100 assurance policy, payable in 40 years or at death, and the corresponding number of years it has been in force :

Number of years						Surrender values	
						£	s.
2	-	-	-	-	-	—	18
4	-	-	-	-	-	3	2
6	-	-	-	-	-	5	10
8	-	-	-	-	-	8	2
10	-	-	-	-	-	10	16
15	-	-	-	-	-	18	14
20	-	-	-	-	-	28	4
25	-	-	-	-	-	39	14
30	-	-	-	-	-	54	4

Draw a graph shewing the relation between the surrender value and the number of years in force. From the graph, determine (i) the surrender value of a £100 policy which has been in force for 22 years, (ii) the number of years a £500 policy must be in force to have a surrender value of £85.
 (L.Ch.C.)

CHAPTER XX

MECHANICAL DEVICES FOR COMPUTATION

20.1. The Need for Calculating Machines.

WHILST a sound knowledge of the basic principles of arithmetic is essential for commercial computation, yet the vast number of long calculations to be made quickly and accurately, which is to-day demanded by large business undertakings, has led to the development of mechanical devices for carrying out much of this work. It is an interesting test to record the time taken to work out on paper such typical practical calculations as the following : *

- (i) 3856 articles at 9s. $4\frac{1}{2}$ d. per gross.
- (ii) 7 tons 17 cwt. 1 qr. at £3 3s. 4d. per ton.
- (iii) Cost of one article when 47,362 cost £22,990 6s. 1d.
- (iv) 532 kilos at 3s. $1\frac{1}{2}$ d. per lb. + 5%. (1 kilo = 2.2 lb.)
- (v) S.I. on £570 6s. 3d. for 224 days at $3\frac{1}{4}$ % per annum.

Now compare the time taken to find each answer on paper with the following times : (i) 7 seconds, (ii) 4 seconds, (iii) 6 seconds, (iv) 11 seconds, (v) 5 seconds. Yet these were the actual approximate times taken by an experienced operator on a suitable calculating machine. It will thus be obvious that, to one whose daily work is to make many such calculations, the time saved by the use of a machine is considerable. Further, the work is more interesting and the drudgery of continuous paper calculation is reduced to a minimum.

So far as is known, the oldest machine directly performing the operations of addition and subtraction was invented in 1642 by Pascal. It is also recorded that in 1850 Thomas of Colmar made an Arithmometer in which numbers were inscribed on cylinders rotated by trains of cog-wheels.

* Answers are given on page xiv at the end.

The principle underlying the mechanism of many machines lies in the fact that the numbers 0, 1, 2, ... 9 are inscribed on circular discs, and when one disc has been rotated through 10 places the next disc to the left is rotated through one place.

20-2. The Ten-key Adding Machine.

Practically all commercial calculation may be reduced fundamentally to *addition* and *subtraction*, for multiplication is but repeated addition, and division is repeated subtraction. Hence, the modern calculating machine is designed primarily for carrying out mechanical addition and subtraction accurately.

The type of machine frequently used and easy to manipulate has only ten keys, labelled 0, 1, 2, ... 9. The **Facit** and **Sundstrand** may be cited as representative of this handy pattern of machine.

In addition to the ten keys, the **Facit** has three long narrow registering panels in which the figures are designed to appear. One of these panels is for setting, one for the product or desired result, and the third for the multiplier. The product and multiplier registers are in line and fitted into a moveable carriage controlled by shift keys like those of a typewriter. When the carriage is in the extreme left position, fixed arrows point to the first position in each of the product and multiplier registers. These serve to indicate the units' digits. The multiplier register records the number of complete turns given to the operating handle. Each register is fitted with a lever for removing or "clearing" any figures shewn in it.

As an example of the use of the machine, suppose it be required to find the sum of 896,324 ; 78,519 ; 639,102 and 93,617. The following operations are necessary.

First ensure that the carriage is in the extreme left position so that the arrows point to the first position on the right in each register, then set 896,324 in the setting register by depressing the appropriate keys just as though the number were to be typed. Turn the operating handle once in the *positive* direction and the

number in the setting register will appear in the product register and, at the same time, 1 will shew in the multiplier register, as indicated in Fig. 34. Now "clear" the setting register and set

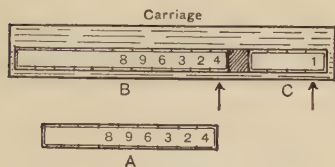


FIG. 34.

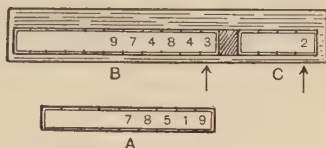


FIG. 35.

the next number, 78,519. Turn the operating handle once in the positive direction; the numbers 974,843 and 2 then appear in the product and multiplier registers respectively. This is indicated in Fig. 35. Thus,

$$896,324 + 78,519 = 974,843.$$

For each of the remaining numbers the process is repeated until finally the carriage registers shew 1,707,562 and 4 respectively, i.e. $896,324 + 78,519 + 639,102 + 93,617 = 1,707,562$.

The 4 in the multiplier register indicates that the operating handle has been turned in the positive direction 4 times.

For subtraction, the process is similar, with the exception that the operating handle is turned in the *negative* direction, i.e. *backwards*.

20·3. The Process for Multiplication.

It will be obvious from Section 20·2 that, when any number N has been set, one turn in the positive direction of the operating handle will put N in the product register; two turns will give $N + N$ or $2N$; three turns, $2N + N$ or $3N$ and, generally, n turns in the positive direction will give the product $N \times n$. When n is greater than 9, it is not always necessary, however, to turn the operating handle n times. Two other methods are available.

Suppose 3478 is to be multiplied by 57. Set 3478 and turn the handle *seven* times in the positive direction; the numbers 24346

and 7 will appear immediately in the product and multiplier registers respectively, as indicated in Fig. 36. This means that 7 positive turns of the handle have produced 7 times 3478, i.e. 24,346.

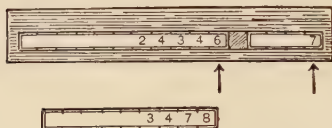


FIG. 36.

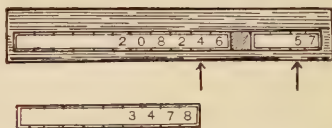


FIG. 37.

Now depress the appropriate key so that the carriage is moved one place to the right; the arrows then point to the tens' digits. Five further positive turns of the handle will then give 198246 in the product register and 57 in the multiplier register, as indicated in Fig. 37.

The reasons for this manipulation may readily be understood by writing down the partial products in the usual way :

$$\begin{array}{r}
 24346 = 3478 \times 7 \\
 17390 = 3478 \times 50 \\
 \hline
 198246 = 3478 \times 57
 \end{array}$$

Had the carriage not been moved one place to the right after multiplying by 7, the five additional turns would have given $24346 + 17390 = 41736$, i.e. $7 + 5$ or 12 times 3478. Hence, in passing from one digit in the multiplier to the next on the left, it is essential to move the carriage one place to the right.

20·4. Short-cutting.

A reduction in the number of turns of the operating handle may often be effected by a process known as **Short-cutting**. Theoretically, the method depends upon a partial decomposition of the multiplier. For example, to multiply by 9 requires 9 positive turns of the handle, but if 9 be considered as $10 - 1$, multiplication by 10 involves one positive turn, with the carriage in the appropriate position, and one negative turn for multiplication by 1. The turn

in the negative direction is necessary as the product by 1 has to be *subtracted* from that by 10. Thus there is a saving of 7 turns. A saving for each of digits 8, 7, 6 may likewise be effected. For a multiplier like 4283, the number of turns by short-cutting would be $+4+2+(1-2)+3$, the signs $+$ and $-$ indicating positive and negative turns respectively. Disregarding directions, the numerical total of the turns needed is 12, whilst, without short-cutting, the total number would be $4+2+8+3=17$.

When a multiplier contains a group of digits together, each of which is greater than, or equal to, 5, short-cutting is advantageous. For instance, a multiplier like 14863 may be considered as

$$14003 + 860 = 14003 + (1000 - 140) = 15003 - 140.$$

Hence, the number of turns required is $+1+5+3-1-4$, making a numerical total of 14. Without short-cutting, the total would be $1+4+8+6+3=22$, i.e. the sum of the digits.

Similarly,

$$35872 = 30002 + 5870 = 30002 + (10000 - 4130) = 40002 - 4130,$$

so that the required number of turns is $+4+2-4-1-3$, giving a numerical total of 14, as compared with $3+5+8+7+2$ or 25 without short-cutting.

After much practice, the theoretical decomposition becomes unnecessary and effective short-cutting visualised almost instinctively.

20.5. Manipulation with Decimal Fractions.

When the numbers involve decimal fractions, the process is precisely similar to that used with whole numbers. The positions of the decimal points are indicated on some machines by moveable pointers which slide on fixed graduated rods, one attached just beneath the setting register and the other above the product and multiplier registers. The numbers may then be dealt with as whole numbers, but it must not be forgotten that, if there are m and n digits, including ciphers, after the decimal points in the

multiplicand and multiplier respectively, then the product must have $(m+n)$ digits, including ciphers, after the decimal point. If all these digits are not required, the pointers may previously be adjusted to indicate the desired approximation.

20-6. The Process for Division.

Considerations of space will prevent more than the briefest sketch of the methods of manipulation used for division.

In the first, called the "tear-down" method, the ordinary processes used in long division are followed. As an example, suppose 91513·14 has to be divided by 365, correct to two places of decimals. The carriage is first moved to its extreme right position. By appropriate setting, the dividend and divisor are made to appear on the extreme left of the product and setting registers respectively, so that the first digits 9 and 3 are in a line. The operating handle is then turned backwards—each turn subtracting 365 from the number formed by the first three digits of the dividend—until 915 has been replaced by the first number less than 365. Two turns will suffice for this and the readings at each turn in the product, multiplier and setting registers are :

9151314	0	5501314	1	1851314	2
365		365		365	

The carriage is next shifted one place to the left and the above operation repeated ; the readings appear as follows :

1851314	20	1486314	21	...	26314	25
365		365			365	

The whole process is repeated until the only number left in the product register is the greatest less than 365. The resulting readings in the product and multiplier registers are

26314	25
764	2507
34	25072

So far the decimal point has not been mentioned. When it is

known how many decimal places are required in the quotient, it is easy to determine at a glance how many digits in the quotient must be found.

Hence, $91513.4 \div 365 = 250.72$ correct to 2 places, since the final remainder shewn, i.e. 34 is less than half 365.

The product register usually contains ten places which, in many machines, show ciphers until numbers are set. As the above dividend contains 7 digits, the final remainder shewn above might appear as 34000. Further digits in the quotient could therefore be easily found if necessary.

Passing now to a second method, let Q be the quotient when D is divided by d ; then

$$\frac{D}{d} = Q \quad \text{or} \quad D = d \times Q.$$

Hence, it is possible by the methods of multiplication to find Q as the number which multiplies d to give D .

As an illustration, suppose it be required to divide 3453 by 73, correct to one place of decimals. Shift the carriage to its extreme right position and set the divisor 73. Turn the handle in the positive direction as many times as will put the largest multiple of 73 less than 345 in the product register. One turn gives 73; two turns give 146; three, 219; four, 292; and five 365, which is too large. Hence, four turns, giving 292, will be sufficient. Shift the carriage one place to the left and repeat the process. Seven positive turns will give 3431 in the product register and 47 in the multiplier register. Shift the carriage again one place to the left and then it will be found that three positive turns will give 34529 which, remembering that this number is really 3452.9, is very near to 3453. It will be obvious that the next figure in the quotient will be 0, so that, from the multiplier register where the number 473 appears, the quotient is actually 47.3, correct to one place. When the number of digits in the quotient is obvious, there is really no need to shift the carriage initially to its extreme right position, but in general it is better to do so to obviate any error

that may have been made in estimating mentally the number of digits in the quotient.

A third method for carrying out division consists in finding, from a reliable table of reciprocals, the value of $1/d$; then the product of this value and D will give Q .

Reverting to the above example, the value of $1/73$ is found to be 0·01370, correct to four places. Now determine the product of 3453 and 0·01370. By the usual process on the machine, the product of 3453 and 137 is found as 473061, so that, correct to one place, $3453 \div 73 = 47\cdot3$.

Division by this method is really done by the table of reciprocals; hence it can hardly be claimed as a satisfactory or direct machine method.

In the absence of a table of reciprocals, the value of $1/d$ might be found on the machine, but there would be no advantage in this, for, instead of dividing unity by d and then multiplying D by the quotient, it would be quicker to divide D by d directly by another method. The method of reciprocals is, in general, only useful when several divisions by the same divisor have to be carried out.

For further information the reader is referred to the English edition of *Modern Machine Calculation* prepared by Dr. J. L. Comrie and Dr. H. O. Hartley (Scientific Computing Service, Ltd., 23, Bedford Square, London, W.C.1.)

20·7. Machines driven electrically.

To replace the manual effort necessary for the continual rotation of the operating handle, electrical power has been utilised for this purpose. A motor, controlled by a switch key or bar, is fitted inside the machine and is supplied with current by a flexible cable which can be plugged into a lamp-holder. This use of electricity as a motive power only gives a slight advantage to the operator and, as a consequence, further developments have been made to reduce the actual work of manipulation. This has been done in many ways, but probably the most striking is that of the automatic multiplier.

In the **Mercedes** machine two groups of keys are used for setting the multiplicand and multiplier respectively ; then, by depressing a special multiplier key, the product appears in the appropriate register. It is possible also, on this machine, to find a continued product, to obtain the sum of a number of products, to determine the power of a number and to carry out the process of division automatically.

Another well-known electric machine is the **Monroe Calculator**, which adds, subtracts, multiplies and divides automatically, giving immediate and accurate results.

20-8. Key-driven Calculators.

When considerable addition and subtraction have to be done quickly and continuously, special machines are desirable. In some of these, where no setting is done, the mechanical lever control of the keyboard is replaced by electrical contacts. As illustrations of this type of machine, it should suffice to mention two only.

In the **Burroughs Electric Duplex Calculator**, shewn in Fig. 38,

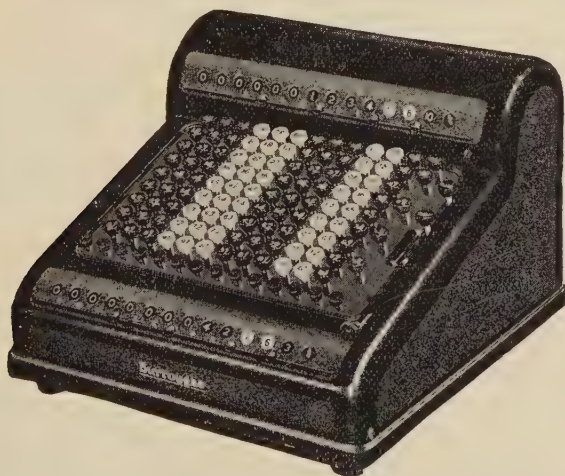


FIG. 38.—Burroughs Electric Duplex Calculator.

(Reproduced by the courtesy of the makers, the Burroughs Adding Machine Ltd.)

addition is automatically carried out when the keys are depressed. Two accumulating registers are provided, one for giving individual totals or results of calculations, the other for shewing the grand total.

A special feature of this machine is its direct subtraction mechanism, by which amounts in one register can be subtracted from amounts in the other register by touching a single key.

Another very efficient machine is the **Felt and Tarrant Electric Comptometer** shewn in Fig. 39.



FIG. 39.—Electric Comptometer.

(Reproduced by the courtesy of the makers, Messrs. Felt and Tarrant, Ltd.)

In the controlled-key type not only is addition automatically performed, but also a controlling mechanism is fitted which provides automatic safeguards against operating errors caused by faulty key strokes. This is a great advantage when high-speed accuracy is essential.

20-9. Combination Typewriter Calculators.

These machines, generally known as Typewriter Accounting Machines, combine the automatic features of an electrically operated calculator with a standard typewriter. They are specially adapted for the speedy preparation of accounts, and not only are the totals calculated, but, also, by the depression of a single key, are printed. Burroughs make several models with special devices for particular kinds of work. The Multiple-Total Machine is shewn

in Fig. 40. This is designed for accounting, distribution, statistical, payroll and other tabulating work which requires full description. On these machines several records can be made at the same time and a large number of totals automatically calculated.

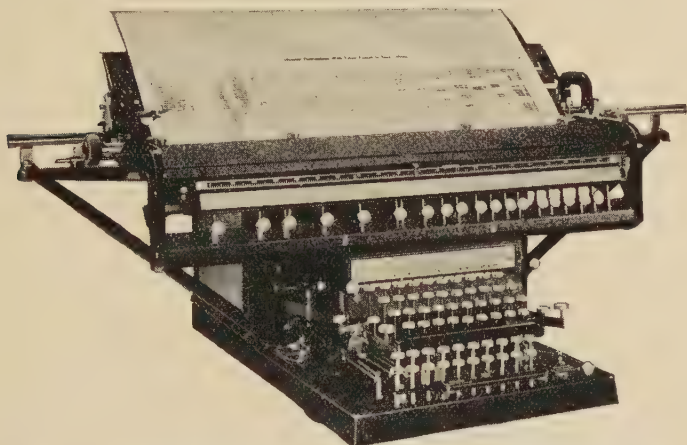


FIG. 40.—Burroughs Multiple-Total Typewriter Accounting Machine.

(Reproduced by the courtesy of the makers, the Burroughs Adding Machine Ltd.)

20·10. Conclusion.

In this brief sketch it has been possible to mention only a few well-known machines to illustrate special features and equipment. It must be understood, however, that there are many varying types of calculating machines on the market designed for specialised work in the business world. It is impossible to deal with these here, for not only would it be out of place in a book on arithmetic, but also an adequate treatment would demand a volume to itself. Indeed, the chief aim of this short chapter is to indicate to the commercial student the importance of modern mechanical means of calculation, and to explain briefly the main principles underlying the simple manipulation of such calculating machines. It must be emphasised that a sound knowledge of the fundamentals

of arithmetic is essential for an intelligent and accurate use both of tables and mechanical calculators, which are two indispensable tools of the modern business house.

TYPICAL EXAMINATION PAPERS

(These are not reprints of actual examination papers, but, in order to make up representative papers of particular examining bodies, the following papers consist of questions either selected from or founded upon questions actually set.)

A. PAPERS IN ELEMENTARY COMMERCIAL ARITHMETIC

(On Part I, Chapters I-XII)

I. THE LONDON CHAMBER OF COMMERCE

ARITHMETIC—ELEMENTARY STAGE

Time allowed—Two hours. Only eight questions to be attempted.

1. If 1 kilogram is equivalent to 2·2046 lb., express 4867 kilograms in tons, cwt., qr., lb., correct to the nearest lb.

2. Three men, *A*, *B*, *C*, bought a business. *A* subscribed five-twelfths of the capital, *B* £468 and *C* £351. In a year's working a profit of £138 was made, and this was divided amongst the three partners in the ratio of their subscribed capital. Find how much each of them received.

3. A draper bought seven dozen pairs of stockings for £23 9s. and, on selling all of them, he made a profit of $12\frac{1}{2}$ per cent. on the total selling price. He sold 4 dozen pairs at 6s. 11d. per pair; what was the selling price per pair of the remaining 3 dozen?

4. A man borrowed from his bank £500 for eight months ending on December 31st. At the end of five months he repaid £372, and the remainder, with interest, on December 31st. If this final payment was £137 0s. 3d., find the rate of interest charged per cent. per annum by the bank.

5. The floor of a rectangular room measures 15 ft. by 13 ft. 6 in. The ceiling is made up of two parts, one parallel to the floor and measuring 8 ft. 7 in. by 13 ft. 6 in., and the other sloping from a height of 8 ft. to a height of 5 ft. Calculate the cost of covering the whole ceiling with paper at 1s. 2d. per square yard.

6. Calculate, to the nearest penny, the cost of 87 tons 17 cwt. 46 lb. at £5 13s. 2d. per ton.

7. A man went for a holiday tour and visited three places X , Y , Z . At X he stayed 6 days at a cost of 14s. 7d. per day; at Y he remained 15 days at a cost of 12s. 10d. per day. The total cost of his visit to Z was £9 12s. 6d., and the average daily expense of the whole tour was 13s. 6d. How many days did he stay at Z and what was the cost there per day?

8. (i) To how many places of decimals does the value of $\sqrt{0.629}$ agree with that of $\frac{23}{9}$?

(ii) If $\pi = 3.1416$, calculate the value of $10 - \pi^2$, giving the result correct to three decimal places.

9. A house costs £819; at what weekly rent must it be let in order to make $6\frac{1}{4}$ per cent. per annum on the cost, after allowing one-eighth of the rent to be spent on repairs?

10. Two dozen cylindrical tins, each $3\frac{1}{2}$ inches in diameter and $4\frac{1}{2}$ inches high, are packed in a rectangular box made to hold them, so that there are two layers, each having 4 tins along one side and 3 along the other in an upright position. Calculate the volume of the air space in the box, taking $\pi = 3\frac{1}{7}$.

II. THE ROYAL SOCIETY OF ARTS EXAMINATIONS

ARITHMETIC. STAGE I—ELEMENTARY

The paper is divided into two parts: MENTAL and WRITTEN

PART I.—MENTAL ARITHMETIC

Time allowed—30 minutes. No calculations on paper allowed.

1. Write down the totals of (i), (ii) and (iii):

	£	s.	d.		£	s.	d.		£	s.	d.
(i)	462	15	2	(ii)	692	5	4	(iii)	962	13	9
	1067	13	10		97	4	5		837	14	6
	29	18	7		6342	18	9		3571	12	8
	4361	14	3		86	16	10		87	18	10
	879	5	11		582	3	6		1672	13	5
	297	8	5		2735	17	7		784	6	11
	1683	13	3		856	14	8		4672	5	8
	769	11	9		432	9	1		439	17	9

2. Write down in £. s. d. to the nearest penny:

(i) £4.837, (ii) £7.183.

3. Express 17s. $7\frac{1}{2}$ d. as an exact decimal of £1.
4. What part of a guinea is $5\frac{1}{4}$ per cent. of £10?
5. What is the cost of 2 lb. 10 oz. at 1s. 8d. per lb.?
6. Give the cost of 49 yards at $11\frac{3}{4}$ d. per yard.
7. How many lengths, each of $3\frac{1}{4}$ yards, can be cut from 52 yards?
8. Give the value of $\frac{7}{20}$ of $(\frac{1}{3} + 1\frac{4}{7})$.
9. Write down in shillings and pence the interest on £218 for one month at $2\frac{1}{2}$ per cent. per annum.
10. An article is bought for 3s. 8d. and is afterwards sold for 4s. 7d.; express the gain as a percentage of the selling price.

PART II.—WRITTEN. *Time allowed: $1\frac{1}{2}$ hours*

1. Calculate the weekly wage bill for a factory in which are employed 236 workmen earning 1s. $10\frac{1}{2}$ d. per hour and 8 foremen earning 2s. $3\frac{1}{2}$ d. per hour, if each man works 45 hours in the week.
2. The total weight of a lorry fully loaded with 83 bags of sand is 9 tons 15 cwt. 3 qr. 1 lb. The lorry alone weighs 2 tons 18 cwt. 2 qr. 18 lb. Find the average weight of each bag of sand.
3. Calculate the necessary height of a rectangular can $10\frac{1}{2}$ inches long by $7\frac{1}{2}$ inches wide which is to have a capacity of $2\frac{1}{2}$ gallons, taking 277·2 cubic inches to a gallon.
4. A man borrows £657 for 45 days at $3\frac{1}{4}$ per cent. per annum. Find, to the nearest penny, the sum he must repay at the end of the period of the loan.
5. A grocer bought one cwt. of tea for £9 12s. 6d. and sold it at 2s. $3\frac{1}{2}$ d. per lb. Find his percentage gain on (i) the selling price and (ii) the cost price.
6. A traveller, on going to Paris, changed £35 into francs when the rate of exchange was 176·4 francs to £1. In Paris he spent 2699·1 francs, and then, on returning, changed the remainder into English money at an exchange rate of 178·2 francs to £1. How much English money did he receive?
7. The area of a square field is 2·09 acres and the length of one side is 91·96 metres. Calculate the number of square metres equivalent to one square yard.

8. A bill of exchange for £1387, drawn on April 10th at three months, is discounted on May 26th at $3\frac{3}{4}$ per cent. per annum. How much did it realise?

III. UNION OF LANCASHIRE AND CHESHIRE INSTITUTES

GENERAL COMMERCIAL AND CLERICAL COURSES

FIRST YEAR

COMMERCIAL ARITHMETIC—PAPER I (MENTAL)

Time allowed—Twenty minutes

1. Write down (i) the horizontal totals, (ii) the vertical totals, and (iii) the grand total of the following :

£	s.	d.	£	s.	d.	£	s.	d.
156	18	2	37	15	9	652	14	9
97	16	10	168	17	6	38	3	10
537	3	5	334	2	1	71	18	7
41	2	11	78	18	7	384	12	2
329	17	4	531	15	10	135	6	10
116	15	7	68	16	2	78	4	11
85	17	3	194	13	6	251	14	1

2. Write down the answers to the following :

- (i) Cost of 57 articles at 4s. 5d. each.
- (ii) 9s. $7\frac{1}{2}$ d. as an exact decimal of £1.
- (iii) Discount on £2 4s. 9d. at 2d. in the shilling.
- (iv) 562×99 .
- (v) $73 \div 125$ as a decimal.
- (vi) The percentage gain of $2\frac{1}{4}$ d. on 3s. 9d.

3. Give the answers to the following calculations :

- (i) £2.4671 in £. s. d. to the nearest penny.
- (ii) Interest on £16 for 5 months at $3\frac{3}{4}$ per cent. per annum.
- (iii) Which is the cheaper rate—£2 13s. per ton or 2s. 8d. per cwt.?
- (iv) Cost of 5 eggs at 2s. 9d. per dozen.
- (v) 6 per cent. of £2 18s. 4d.
- (vi) $4\frac{1}{2}$ yards at 2s. 11d. per yard.

PAPER II

Time allowed—Two hours ten minutes.

Answer Questions 1 and 2 and four other questions.

1. Simplify : (a) $\frac{4\frac{2}{9} - 1\frac{13}{18}}{2\frac{1}{5} + (\frac{4}{7} \times 2\frac{1}{3})}$ of £7 10s. 2 1
 (b) $\frac{14.83 \times 0.362}{37.34}$, correct to three decimal places.

2. A man borrows £600 for 4 years at $3\frac{1}{2}$ per cent. per annum compound interest. At the end of each of three years he repays £170. What final payment, to the nearest penny, must he make at the end of the fourth year to clear the debt?

3. A coal dealer bought four lots of coal for which he paid £20 2s., £17 10s., £26 1s. 4d., £31 13s. 4d. respectively. The respective prices per ton were : £1 2s. 4d., £1 5s., £1 2s. 8d., £1 5s. 4d. Calculate the average price per ton that he paid for the whole of the coal.

4. A trader buys an article for £5 4s. 2d. and sells it to gain 14 per cent. on the cost. Later the cost price of a similar article is raised and, as a consequence, the trader increases his selling price by 10s. 5d., thus making a profit of 14 per cent. on the selling price. Find how much the cost price was raised.

5. Calculate, to three significant figures, the number of acres equivalent to 1 hectare, having given that 1 square kilometre = 100 hectares and 1 yard = 0.9144 metre.

6. The weight of 197 castings is 36 tons 1 cwt. 18 lb. Find

- (i) the average weight of each casting, and
 (ii) its value, if the material is worth £1 3s. 4d. per cwt.

7. A creditor receives 13s. 1d. in the £ on a debt of £612 and 8s. 11d. in the £ on a debt of £1428. Calculate the average amount in the £ he receives from the two debts combined.

8. A bill for £1825 is discounted 48 days before it is due, the discount being £6 12s. Find the rate per cent. per annum of interest allowed.

N.B.—In these papers questions may be given involving easy logarithms and simple graphs.

B. PAPERS AT AN INTERMEDIATE STAGE

I. THE CHARTERED INSTITUTE OF SECRETARIES

INTERMEDIATE EXAMINATION. COMMERCIAL ARITHMETIC

Six questions only to be attempted, three from each section.

Logarithmic tables may be used, if desired, for any question.

Time allowed—Two hours.

SECTION A

1. During a certain month, a trader bought

- (i) 182 articles for £909 4s. 7d. and sold them all at £5 16s. each ;
- (ii) 267 articles at £2 17s. 6d. each and sold the lot for £842 3s. 3d. ;
- (iii) 3835 articles at £13 15s. per 100 and sold all of them at 4s. 9d. each.

His expenses for the month were 6 per cent. of the total cost of the above goods. Find his actual profit and express it as a percentage of the total outlay.

2. A shopkeeper marks a certain class of goods 32 per cent. above cost price, but he allows cash customers a discount of 12 per cent. and credit customers $2\frac{1}{2}$ per cent. off the marked price. If, during a given period, the numbers of his cash and credit customers are in the ratio of 5 : 6, calculate what percentage of profit he makes on the cost price.

3. The gross profit of a company for May was £9860, which represented 29 per cent. on sales. In June the profit was £12,971, representing 35 per cent. on sales. Find (i) the percentage increase in the sales of June on those of May, and (ii) the percentage of the total profit on the sales for the two months.

4. A bill of exchange for 102,784 francs was accepted on March 6th at four months. It was discounted in Paris on April 25th at $2\frac{3}{4}$ per cent. per annum. The proceeds were sent to London, the exchange rate then being 176 francs to £1. Find the amount received in London.

SECTION B

5. A loan of £12,500, together with compound interest at 5 per cent. per annum, is paid off in five equal annual instalments, the

first payment being made at the end of the first year. Calculate, to the nearest £, the amount of each payment.

6. The purchase of a $4\frac{1}{2}$ per cent. stock yields 4·8 per cent. on the money invested. From such an investment, an annual income of £171 12s. is obtained after income tax at 7s. in the £ has been deducted. Find (i) the price of the stock and (ii) the sum invested.

7. The capital needed for a business is provided by three partners, *A*, *B*, *C*. *A* subscribes £4459, *B* £3773, *C* £3087. It is agreed that, out of the profits at the end of the year, *A* shall receive £800 as manager, *B* shall receive £600 as secretary, and £500 shall be put into a reserve fund. Any profit remaining shall be divided amongst the three partners in proportion of capital subscribed. If the year's working yielded a profit of £3121, how much, including salary, did each partner receive?

8. To pay off a mortgage of £1432 at the end of twelve years, a man invests twelve equal amounts, one at the end of each year, at $3\frac{1}{2}$ per cent. per annum compound interest. Calculate, to the nearest shilling, each amount.

II. THE LONDON CHAMBER OF COMMERCE

CERTIFICATE EXAMINATION. ARITHMETIC

Candidates are to attempt either Question 1 or Question 2 and no more than six other questions.

Time allowed—Two hours

1. (i) To how many places of decimals does $\frac{117}{253}$ agree with $\sqrt{0\cdot21386}$?

(ii) The week's wages of 5879 men are £20,160 1s. 5d. Calculate the average weekly wage of a man. If an additional 73 men were employed at the same average rate, what further sum would be needed?

2. The total value of goods imported into Britain for the six months ending December 31st, 1936, was £449,546,737. The values for five months were as follows : July £68,731,020 ; August £66,057,087 ; October £80,539,176 ; November £78,671,360 ; December £83,656,566. What was the value for September?

The total value for the twelve months ending December 31st, 1936, was £852,397,579 ; what was the value for the six months ending June 30th?

Calculate, to two places of decimals, the percentage increase in the total value for the period July-December on that for the period January-June. Calculate also the percentage increase of the highest monthly value for the period July-December on the average monthly value for the whole year.

3. A draper bought 3 dozen pairs of stockings at two guineas per dozen pairs and 2 dozen pairs at £2 5s. per dozen pairs. On the sale of all these stockings, he made a profit of 20 per cent. on the selling price, whilst on the cheaper quality alone he made a profit of 16 per cent. Find the selling price per pair of the better quality stockings.

4. On what sum of money is the compound interest for three years at $3\frac{3}{4}$ per cent. per annum £249 2s. 3d.?

5. The rateable value of a town is £298,775, and the rate levied on this value is 11s. 3d. in the £. The next year, in consequence of a re-arrangement of boundaries, the rateable value was increased by 8 per cent. and the rate levied was reduced. The sum raised by this rate was, however, greater than that raised the previous year by £5377 19s. Find the reduction in the rate.

6. A man holds investments in two stocks, one $4\frac{1}{4}$ per cent. at $107\frac{1}{4}$, and the other $3\frac{1}{2}$ per cent. at $97\frac{1}{2}$. The annual incomes derived from these are as 5 : 7, and after deducting income tax at 5s. in the £, his total annual income from both investments was £295 16s. How much was invested in each stock?

7. The net receipts for income tax in Great Britain are given below for ten years :

Years	Tax in Millions of pounds			
1926-27	-	-	-	230.1
27-28	-	-	-	253.5
28-29	-	-	-	237.3
29-30	-	-	-	237.9
30-31	-	-	-	255.3
31-32	-	-	-	288.4
32-33	-	-	-	250.6
33-34	-	-	-	228.6
34-35	-	-	-	229.2
35-36	-	-	-	237.4

Represent these statistics graphically and, from the graph, determine

- (i) the greatest rate of increase in the tax yield between consecutive years,
- (ii) the estimated yield in 1936-37 if the greatest rate of increase between consecutive years were maintained between 1935-36 and 1936-37.

8. Calculate, in square inches, the area of sheet metal required to make an open cylindrical vessel, 7·2 inches in height, whose capacity shall be one quart, having given that 1 quart = 69·3 cubic inches and $\pi = 3\frac{1}{7}$.

9. The debts in a bankruptcy were £8073, including one to a secured creditor for £465. The assets were £2638, together with the value of the stock. The legal expenses were £237 and a dividend of 10s. 10d. in the £ was paid. Find the value of the stock.

10. Goods purchased in France at 38·5 francs per kilogram were sent to London where a duty of 7s. per cwt. was paid. They were then sold at 3s. $1\frac{1}{2}$ d. per lb. at a profit of 34 per cent. on the selling price. Find the rate of exchange in francs to the £, taking 2·2 lb. to 1 kilogram.

III. THE ROYAL SOCIETY OF ARTS EXAMINATIONS

ARITHMETIC. STAGE II—INTERMEDIATE

Three hours allowed. No logarithms are to be used.

1. Find the cost of 18 tons 16 cwt. of rice at 38s. $9\frac{1}{2}$ d. per cwt.
2. The invested capital in a company consists of £8,602,497 at 3 per cent., £732,456 at $3\frac{1}{4}$ per cent., £3,467,852 at $4\frac{1}{2}$ per cent., and £3,642,770 at 5 per cent. Calculate, to the nearest penny, the yearly interest payable by the company on these investments.
3. The wages paid by a firm are 52 per cent. of the total cost of production, on which cost the sales yield a profit of $27\frac{1}{2}$ per cent. When wages are raised by five per cent., other costs remaining the same, by how much per cent. must the sales be increased so that the same percentage profit on the total cost may be made?
4. Find the amount, to the nearest penny, of £356 11s. 3d. for $2\frac{1}{2}$ years at 4 per cent. per annum compound interest payable half-yearly.
5. Find the multiplier, correct to four significant figures, which will convert prices in francs per metre into the equivalent in pence per yard, given that 1 yard = 0·9144 metre and £1 = 178 francs.

6. Find the freight of 35 cases of goods each measuring 4 ft. 6 in. by 3 ft. 9 in. by 2 ft. 8 in. at 14s. 8d. per "ton" of 40 cubic feet.

If the average weight per case is 14 cwt. 16 lb., find the freight per ton weight.

7. When the price of a $6\frac{1}{4}$ per cent. preference £1 share is £1 11s. 8d. including brokerage, how many shares must be purchased to yield an annual income of £247 after income tax at 7s. in the £ has been deducted?

Find also the cost of the shares.

8. In the case of a bankrupt it was estimated that the sum available for distribution among the creditors would be £8791, which would provide them with 12s. $3\frac{1}{2}$ d. in the £. When the assets were realised and all expenses paid, it was found, however, that only £7897 was available to pay the creditors. How much in the £ did they actually receive?

9. A bill of exchange was discounted 16 weeks before it was legally due at $4\frac{1}{4}$ per cent. per annum and realised £1309 14s. 7d. What was the face value of the bill? Take 52 weeks to a year.

10. It is required to construct a cylindrical tank on a circular base to hold 726 gallons of petrol. Calculate the height of the tank, correct to the nearest tenth of an inch, given that the internal diameter = 5 ft. 6 in., 1 gallon = 277·274 cub. in. and $\pi = 3\cdot1416$.

IV. UNION OF LANCASHIRE AND CHESHIRE INSTITUTES

SENIOR COMMERCIAL COURSES. SECOND YEAR

COMMERCIAL ARITHMETIC—PAPER II

Time allowed—2 hours 10 minutes

Six questions only to be answered. Squared paper and Mathematical Tables are supplied.

(Note.—Paper I consists of a Mental Test similar to that shewn on page 320, although slightly more difficult. Twenty minutes are allowed for this test.)

1. Find, by the use of logarithms,

$$(i) \sqrt{\frac{18\cdot93 \times 0\cdot04797}{3\cdot732 \times (1\cdot876)^3}}$$

(ii) The price per yard in shillings and pence equivalent to 41·37 francs per metre, given that £1 = 178 francs and 1 metre = 39·37 inches.

2. The profits of a business for eight consecutive years were : £413,624 ; £415,731 ; £407,865 ; £398,876 ; £416,429 ; £409,873 ; £417,122 and £416,502 respectively. Find what percentage increase the eighth year's profit was of the average for the previous seven years.

3. It is necessary to reduce the weight by $4\frac{1}{4}$ per cent. of a rectangular plate 16·8 in. by 9·44 in. and uniformly thick by drilling a circular hole through it. Calculate, preferably by use of tables, the diameter of the hole, taking $\pi = 3\cdot14$.

4. Three bills for £584, £438 and £657 respectively were discounted at $3\frac{3}{4}$ per cent. per annum by a banker at the same time. The bills were legally due in 65, 45 and 25 days respectively from the time they were discounted. Calculate the total discount allowed by the banker.

5. Find (i) the amount for 23 years and (ii) the compound interest for the 23rd year when £352 4s. is invested at $4\frac{1}{4}$ per cent. per annum compound interest, having given that

$$\begin{array}{ll} \log 3\cdot522 = 0\cdot5467894, & \log 9\cdot1736 = 0\cdot9625397, \\ \log 1\cdot0425 = 0\cdot0180761, & \log 8799\cdot614 = 3\cdot9444636. \end{array}$$

6. *A* with £5427 capital and *B* with £2345 formed a partnership. It was arranged that each partner should receive interest on capital at 5 per cent. per annum, that *B* should take 20 per cent. of the net profit for managing the business, and that the remainder of the net profit should be shared between *A* and *B* in proportion to capital. If the first year's net profit was £761 5s., find the income, inclusive of interest, derived from the business by each partner.

7. A man invests part of £7740 in a $3\frac{3}{4}$ per cent. stock at $92\frac{1}{2}$ and the remainder in a $5\frac{1}{4}$ per cent. stock at $113\frac{3}{4}$. His total annual income from these two investments yields $4\frac{1}{2}$ per cent. of the total sum invested. Find how much he invests in each stock.

8. A householder is charged for electricity consumed an amount made up of two parts, one being a fixed sum and the other being proportional to the number of units consumed. When 292 units are used the charge is £1 16s. ; when 476 units are used the charge

is £2 7s. 6d. Draw a graph shewing the charges for the numbers of units used from 0 to 500, and from it determine

- (i) the charge when 436 units are consumed, and
- (ii) the fixed sum.

C. PAPERS AT AN ADVANCED STAGE

I. THE LONDON CHAMBER OF COMMERCE

HIGHER COMMERCIAL EDUCATION CERTIFICATE

COMMERCIAL ARITHMETIC

Time allowed—3 hours. Not more than eight questions are to be attempted.

1. In Paris perfume costs 412 francs per litre. When brought into England a duty of $33\frac{1}{3}$ per cent. of the price paid is levied. It is then sold at 4s. 6d. per bottle of capacity one-twelfth of a pint on which a profit of 35 per cent. is made on the selling price. Find the rate of exchange to the nearest franc, in francs to the £. Take one litre to be equivalent to 1.76 pints.

2. In one year the takings of a certain borough's tramways consisted of 1,794,563 penny fares, 893,421 two-penny fares and 413,258 three-penny fares. The expenses were: wages £8543 14s. 3d., maintenance, etc., £5497 7s. 5d., and interest at $4\frac{1}{2}$ per cent. on a loan of £50,000. In addition, a repayment of £2000 was made off the loan, and the remaining receipts were used to reduce the rates of the borough, the rateable value being £287,546. Calculate the actual reduction in the £.

3. The compound interest on £500 for 5 years is £99 16s. Find the compound interest on £625 for 15 years at the same rate per cent. per annum.

4. A tradesman bought goods to the following amounts on the dates specified, one month being allowed in each case for payment: £136 on April 12th; £116 on May 8th; £134 on May 30th; £216 on June 24th. On June 10th he made a payment of £238; on what date must the balance be paid in order that the account may be equitably settled?

5. An article priced at £18 10s. may be purchased by an immediate payment of £3 10s. followed by nine monthly payments

of £1 15s. each, the first to be paid one month after purchase. Calculate the rate per cent. per annum simple interest charged.

6. A father left £11,067 to his two sons so that the elder should receive his share in three years' time and the younger in seven years' time, the two shares being equal when received. Reckoning compound interest at 4 per cent. per annum, find the amount of each share.

7. Machinery to the value of £19,120 was purchased on the following agreement. £1000 to be paid at the time of purchase, and then £1200 to be paid annually until the debt, with interest, was cleared, the first instalment to be paid one year after purchase. Reckoning compound interest at $4\frac{1}{2}$ per cent. per annum, calculate the number of yearly payments necessary.

8. A retailer sells goods at a price by which he gains 28 per cent. profit on that price. The cost price of his goods is reduced later by 8 per cent. and he lowers his selling price by 4 per cent. Calculate his percentage profit on sales under these conditions.

9. A man invests £1003 in a $2\frac{3}{4}$ per cent. stock at $88\frac{1}{2}$; £8916 in a $3\frac{1}{2}$ per cent. stock at 63, and he wishes to invest in a $4\frac{1}{4}$ per cent. stock at $108\frac{1}{2}$ as much money as will yield on the three investments $3\frac{1}{4}$ per cent. of the total capital invested, after deducting income tax at seven shillings in the £. How much must he invest in the $4\frac{1}{4}$ per cent. stock, all charges being included in the prices specified?

10. To endow a scholarship of £90 a year, the first payment to be made a year hence, an insurance company required a sum of £4000. Reckoning compound interest, calculate the rate per cent. per annum charged.

II. THE ROYAL SOCIETY OF ARTS EXAMINATIONS

ARITHMETIC. STAGE III—ADVANCED

Three hours allowed

PART I

1. Calculate the value of a rectangular sheet of lead $31\frac{1}{2}$ inches wide, 15 feet long and $\frac{5}{16}$ inch thick, at £17 1s. 4d. per ton, taking one cubic foot of lead to weigh 712 lb.

2. For manufacturing certain articles, a company was formed

with a capital of £10,846 in ordinary shares and £2500 in 5 per cent. preference shares. In one year 10,422 articles were made at an average cost of £34 13s. 4d. per gross and sold at 11s. 3d. each. From the profit thus made, dividends due to the preference shareholders were paid; £1899 4s. 2d. was used for overhead expenses, etc., and 32 per cent. of the remainder was transferred to the reserve fund. The rest of the profit was then used to pay interest to the ordinary shareholders. What was the rate per cent. of their dividend?

3. A bill of exchange for £912 10s. drawn on April 5th at three months was discounted at $3\frac{1}{2}$ per cent. per annum and realised £908 11s. 3d. On what date was it discounted?

4. A man borrowed £6375 on the understanding that it was to be repaid, with compound interest, at 4 per cent. per annum, in two equal instalments to be paid at the ends of the second and third years respectively after the date of the loan. Without using logarithms, calculate the amount of each instalment.

5. A sphere and a rectangular solid 18·7 inches by 15·4 inches by 13·8 inches have equal volumes. Calculate (i) the diameter of the sphere and (ii) its surface area in square feet. Take the volume of a sphere of diameter d inches to be $0\cdot5236 \cdot d^3$ cubic inches.

6. The liabilities of a bankrupt are £6852. There is one secured creditor whose claim for £573 must be paid in full, and the legal expenses are £132 19s. The total assets amount to £5017; how much in the £ can be paid to the ordinary creditors?

PART II

Three only of these questions are to be attempted.

7. An alloy contains 32 per cent. by weight of copper, 23 per cent. of tin and the remainder zinc. How many lb. of copper must be melted up with one cwt. of this alloy in order that the percentage of copper may be increased to 36 per cent.? Find also the percentage of zinc in the new alloy.

8. Formerly a rate of 12s. 1d. in the £ was needed to meet the annual expenditure of a certain town. Since then, the expenditure met by the rates has increased by 27·1 per cent. and the rateable value of the town has increased by 18·9 per cent. Calculate the rate in the £ now required.

9. A company offers £100 stock on the following terms: £16 at once, £24 after one month, £28 after three months and £32 after five months. What is the equivalent purchase price per £100 stock if paid at once, reckoning discount at $3\frac{3}{4}$ per cent. per annum?

If the stock is a $3\frac{3}{4}$ per cent. stock and the first half-yearly dividend is due one month after payment of the last instalment, calculate the amount of the first dividend per £100 stock, after deducting 7s. in the £ income tax.

10. To purchase a house a man borrowed £1250 at $3\frac{1}{2}$ per cent. per annum compound interest. He agreed to repay the loan with interest in 24 equal half-yearly instalments, the first to be paid six months after the date of the loan. Calculate, to the nearest penny, how much each payment must be, given that

$$\log 10175 = 4.0075344.$$

11. A manufacturer allows his customers a trade discount of 30 per cent. off his catalogue prices and thus obtains a profit of 12 per cent. on the cost of manufacture. Some time later, the cost of manufacture is increased by $6\frac{2}{3}$ per cent. The manufacturer keeps his catalogue prices unchanged, but only allows a trade discount of 25 per cent. off them. Find what percentage profit he makes on these terms (i) on the cost of production and (ii) on his actual selling price.

12. $ABCD$ is a rectangle having $AB = 12.3$ inches and $AD = 14.8$ in. A point P is taken in BC such that $BP = 6.4$ in. and a point Q is taken in DC such that $DQ = 4.6$ in. The five-sided figure $ABPQD$ then generates a solid of revolution about the side AD ; calculate the volume of this solid in cubic feet, correct to three significant figures, taking $\pi = 3.14$.

III. UNION OF LANCASHIRE AND CHESHIRE INSTITUTES

SENIOR COMMERCIAL COURSE—THIRD YEAR

COMMERCIAL ARITHMETIC

Time allowed—2½ hours. Answer Questions 1 and 2 and four other questions. Mathematical Tables and squared paper are supplied.

1. The following table gives a few statistics concerning the value of British imports in 1935 and 1936:

		Value of total imports	Value of imported food
1935	-	£756,040,537	£337,546,662
1936	-	£848,935,895	£364,192,347

Determine (i) the percentage increases in the total imports and the imported food in 1936 on those of 1935, and (ii) the food imports expressed as a percentage of the total imports for each year. Give each result correct to one decimal place.

2. For an importer, a banker discounts a bill for £8005 13s. 4d. at $2\frac{1}{2}$ per cent. per annum which is legally due in 75 days' time. With the amount thus realised, the importer buys two foreign bills, one on Paris for 38,346 francs and the other on Brussels for 184,177 belgas. The exchange on Paris is 176 francs to the £, find the exchange on Brussels in belgas to the £.

3. Bronze used for coinage is an alloy consisting of 95 per cent. by weight of copper, 4 per cent. of tin and the remainder zinc. Calculate the value of one cwt. of such bronze when the prices per ton of copper, tin and zinc are £55, £237 10s. and £41 13s. 4d. respectively.

4. A draper bought fifteen rolls of cloth, each containing one dozen yards. He hoped to sell the whole of this at 8s. 9d. per yard and thus realise a profit of 15 per cent. on the selling price. Actually, however, he was only able to sell twelve rolls at this price, and for the remainder he had to reduce the price to 7s. 6d. per yard. What percentage profit on the total proceeds did he make?

5. When the income tax was increased from 5s. 6d. in the £ to 7s., a man sold his holding in a $4\frac{1}{2}$ per cent. stock at $108\frac{3}{4}$ and invested the proceeds in a $5\frac{1}{4}$ per cent. stock at 91. Find (i) the increase in the percentage yield after payment of income tax, and (ii) the sum invested if the increase in the net income realised by the transfer was £12 5s. 3d.

6. On March 15th a man borrows £1168 and in order to repay the debt with interest at $4\frac{1}{2}$ per cent. per annum he gives two bills, one for £611 legally due on July 18th, and the other to be legally due on September 29th. Find the amount of this second bill.

7. Calculate, to the nearest penny, the amount of an annuity payable yearly for 24 years for which the purchase price is £2500, the first payment to be made one year after purchase. Reckon compound interest at $3\frac{1}{2}$ per cent. per annum.

8. The following table gives the annual premiums payable from the ages specified for an endowment assurance with profits of £100, which matures at the age of 60, or at death, if before that date :

Age				Annual Premium	
				£	s.
26	-	-	-	2	10
28	-	-	-	2	14
30	-	-	-	3	0
32	-	-	-	3	6
33	-	-	-	3	9
35	-	-	-	3	15
36	-	-	-	3	19
38	-	-	-	4	9

Draw a graph connecting age with the premium payable from it ; determine from the graph :

- (i) the annual premium payable from the age of 34, and
- (ii) the age from which an annual premium of £2 17s. would be payable.

DECIMALISATION TABLES

I. Shillings and Farthings as Decimals of £1 to Eight Places

Shillings	£	Farthings	£	Farthings	£
1	0·05	1	0·00104167	25	0·02604167
2	0·10	2	0·00208333	26	0·02708333
3	0·15	3	0·00312500	27	0·02812500
4	0·20	4	0·00416667	28	0·02916667
5	0·25	5	0·00520833	29	0·03020833
6	0·30	6	0·00625000	30	0·03125000
7	0·35	7	0·00729167	31	0·03229167
8	0·40	8	0·00833333	32	0·03333333
9	0·45	9	0·00937500	33	0·03437500
10	0·50	10	0·01041667	34	0·03541667
11	0·55	11	0·01145833	35	0·03645833
12	0·60	12	0·01250000	36	0·03750000
13	0·65	13	0·01354167	37	0·03854167
14	0·70	14	0·01458333	38	0·03958333
15	0·75	15	0·01562500	39	0·04062500
16	0·80	16	0·01666667	40	0·04166667
17	0·85	17	0·01770833	41	0·04270833
18	0·90	18	0·01875000	42	0·04375000
19	0·95	19	0·01979167	43	0·04479167
		20	0·02083333	44	0·04583333
		21	0·02187500	45	0·04687500
		22	0·02291667	46	0·04791667
		23	0·02395833	47	0·04895833
		24	0·02500000		

Use of Table I

Ex. 1. To express 17s. $7\frac{3}{4}$ d. as a decimal of £1.

From the table :

$$17\text{s.} = \text{£}0\cdot85.$$

$$7\frac{3}{4}\text{d.} = 31 \text{ farthings} = \text{£}0\cdot03229167.$$

$$\therefore 17\text{s. } 7\frac{3}{4}\text{d.} = \text{£}0\cdot88229167.$$

Ex. 2. Convert £0.67394851 into shillings and pence.

$$£0.67394851 = £(0.65 + 0.02394851).$$

Now £0.65 = 13s. and £0.02394851 is nearer £0.02395833 than £0.02291667, i.e. nearer 23 farthings than 22 farthings.

$$\therefore £0.67394851 = 13s. + 23 \text{ farthings}$$

$$= 13s. 5\frac{3}{4}d. \text{ to the nearest farthing, or}$$

$$= 13s. 6d. \text{ to the nearest penny.}$$

II. Cwt., Qr., Lb. as Decimals of 1 Ton to Nine Places

Cwt.	Ton	Qr.	Ton	Lb.	Ton	Lb.	Ton
1	0.05	1	0.0125	1	0.000446429	15	0.006696429
2	0.10	2	0.0250	2	0.000892857	16	0.007142857
3	0.15	3	0.0375	3	0.001339286	17	0.007589286
4	0.20			4	0.001785714	18	0.008035714
5	0.25			5	0.002232143	19	0.008482143
6	0.30			6	0.002678571	20	0.008928571
7	0.35			7	0.003125000	21	0.009375000
8	0.40			8	0.003571429	22	0.009821429
9	0.45			9	0.004017857	23	0.010267857
10	0.50			10	0.004464286	24	0.010714286
11	0.55			11	0.004910714	25	0.011160714
12	0.60			12	0.005357143	26	0.011607143
13	0.65			13	0.005803571	27	0.012053571
14	0.70			14	0.006250000		
15	0.75						
16	0.80						
17	0.85						
18	0.90						
19	0.95						

Use of Table II

Ex. Express (i) 13 cwt. 3 qr. 17 lb. as a decimal of 1 ton,

(ii) 2 qr. 23 lb. as a decimal of 1 cwt.

(i) From the table :

$$\begin{array}{rcl}
 13 \text{ cwt.} & = & 0.65 \quad \text{ton.} \\
 3 \text{ qr.} & = & 0.0375 \quad \text{,,} \\
 17 \text{ lb.} & = & 0.007589286 \quad \text{,,} \\
 \hline
 \therefore 13 \text{ cwt. } 3 \text{ qr. } 17 \text{ lb.} & = & \underline{0.695089286} \text{ ton.}
 \end{array}$$

This value may be reduced to as many places as necessary ; thus to four places, it is **0.6951** ton ; to five places, it is **0.69509** ton, and so on.

(ii) Again, from the table :

$$\begin{array}{rcl}
 2 \text{ qr.} & = & 0.025 \quad \text{ton.} \\
 23 \text{ lb.} & = & 0.010267857 \quad \text{,,} \\
 \hline
 \therefore 2 \text{ qr. } 23 \text{ lb.} & = & \underline{0.035267857} \text{ ton} \\
 & = & 0.035267857 \times 20 \text{ cwt.} \\
 & = & \underline{0.70535714} \text{ cwt.}
 \end{array}$$

Use of Table III

Ex. *Express as a decimal of a mile :*

(i) 567 yards and (ii) 57 chains 65 links.

(i) From the table :

$$\begin{array}{rcl}
 500 \text{ yards} & = & 0.284090909 \text{ mile.} \\
 67 \text{ ,,} & = & 0.038068182 \quad \text{,,} \\
 \hline
 \therefore 567 \text{ yards} & = & \underline{0.322159091} \text{ mile.}
 \end{array}$$

(ii) 57 chains 65 links = 57.65 chains = 57.65 \times 22 yards = 1268.3 yards.

Hence, from the table :

$$\begin{array}{rcl}
 1200 \text{ yards} & & = 0.681818182 \text{ mile.} \\
 68 \text{ ,,} & & = 0.038636364 \quad \text{,,} \\
 0.3 \text{ yd.} & = 0.1 \text{ of } 3 \text{ yd.} & = 0.000170454 \quad \text{,,} \\
 \hline
 \therefore 1268.3 \text{ yd. or } 57 \text{ ch. } 65 \text{ links} & = & \underline{0.720625000} \text{ mile.}
 \end{array}$$

III. Yards as Decimals of 1 Mile to Nine Places

Yd.	Mile	Yd.	Mile	Yd.	Mile	Yd.	Mile
1	0·000568182	30	0·017045455	59	0·033522727	88	0·050000000
2	0·001136364	31	0·017613636	60	0·034090909	89	0·050568182
3	0·001704545	32	0·018181818	61	0·034659091	90	0·051136364
4	0·002272727	33	0·018750000	62	0·035227273	91	0·051704545
5	0·002840909	34	0·019318182	63	0·035795455	92	0·052272727
6	0·003409091	35	0·019886364	64	0·036363636	93	0·052840909
7	0·003977273	36	0·020454545	65	0·036931818	94	0·053409091
8	0·004545455	37	0·021022727	66	0·037500000	95	0·053977273
9	0·005113636	38	0·021590909	67	0·038068182	96	0·054545455
10	0·005681818	39	0·022159091	68	0·038636364	97	0·055113636
11	0·006250000	40	0·022727273	69	0·039204545	98	0·055681818
12	0·006818182	41	0·023295455	70	0·039772727	99	0·056250000
13	0·007386364	42	0·023863636	71	0·040340909	100	0·056818182
14	0·007954545	43	0·024431818	72	0·040909091	200	0·113636364
15	0·008522727	44	0·025000000	73	0·041477273	300	0·170454545
16	0·009090909	45	0·025568182	74	0·042045455	400	0·227272727
17	0·009659091	46	0·026136364	75	0·042613636	500	0·284090909
18	0·010227273	47	0·026704545	76	0·043181818	600	0·340909091
19	0·010795455	48	0·027272727	77	0·043750000	700	0·397727273
20	0·011363636	49	0·027840909	78	0·044318182	800	0·454545455
21	0·011931818	50	0·028409091	79	0·044886364	900	0·511363636
22	0·012500000	51	0·028977273	80	0·045454545	1000	0·568181818
23	0·013068182	52	0·029545455	81	0·046022727	1100	0·625000000
24	0·013636364	53	0·030113636	82	0·046590909	1200	0·681818182
25	0·014204545	54	0·030681818	83	0·047159091	1300	0·738636364
26	0·014772727	55	0·031250000	84	0·047727273	1400	0·795454545
27	0·015340909	56	0·031818182	85	0·048295455	1500	0·852272727
28	0·015909091	57	0·032386364	86	0·048863636	1600	0·909090909
29	0·016477273	58	0·032954545	87	0·049431818	1700	0·965909091

For an example of the use of this table see page 336.

FOUR-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	12 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	59 13	17 21 26	30 34 38
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	48 12	16 20 24	28 32 36
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	47 11	15 18 22	26 29 33
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	37 11	14 18 21	25 28 32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	36 10	13 16 19	23 26 29
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	37 10	13 16 19	22 25 29
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	36 9	12 15 19	22 25 28
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	36 9	12 14 17	20 23 26
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	36 8	11 14 17	19 22 25
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	36 8	11 14 16	19 22 24
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	35 8	10 13 15	18 21 23
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	35 8	10 12 15	17 20 23
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	25 7	9 12 14	17 19 21
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	24 7	9 11 14	16 18 21
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	24 6	8 11 13	15 17 19
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	24 6	8 11 13	15 17 19
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	24 6	8 10 12	14 16 18
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	24 6	8 10 12	14 15 17
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	24 6	7 9 11	13 15 17
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	24 5	7 9 11	12 14 16
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	23 5	7 9 10	12 14 15
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	23 5	7 8 10	11 13 15
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	23 5	6 8 9	11 13 14
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	23 5	6 8 9	11 12 14
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	23 4	6 7 9	10 12 13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	13 4	6 7 9	10 11 13
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	13 4	6 7 8	10 11 12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	13 4	5 7 8	9 11 12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	13 4	5 6 8	9 10 12
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	13 4	5 6 8	9 10 11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	12 4	5 6 7	9 10 11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	12 4	5 6 7	8 10 11
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	12 3	5 6 7	8 9 10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	12 3	5 6 7	8 9 10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	12 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	12 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	12 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	12 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	12 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	12 3	4 4 5	6 7 8

FOUR-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
100	0000000	0004341	0008677	0013009	0017337	0021661	0025980	0030295	0034605	0038912
1	43214	47512	51805	56094	60380	64660	68937	73210	77478	81742
2	86002	90257	94509	98756	0103000	0107239	0111474	0115704	0119931	0124154
3	0128372	0132587	0136797	0141003	45205	49403	53598	57788	61974	66155
4	70333	74507	78677	82843	87005	91163	95317	99467	0203613	0207755
5	0211893	0216027	0220157	0224284	0228406	0232525	0236639	0240750	44857	48960
6	53059	57154	61245	65333	69416	73496	77572	81644	85713	89777
7	93838	97895	0301948	0305997	0310043	0314085	0318123	0322157	0326188	0330214
8	0334238	0338257	42273	46285	50293	54297	58298	62295	66289	70279
9	74265	78248	82226	86202	90173	94141	98106	0402066	0406023	0409977
10	0413927	0417873	0421816	0425755	0429691	0433623	0437551	41476	45398	49315
11	53230	57141	61048	64952	68852	72749	76642	80532	84418	88301
12	92180	96056	99929	0503798	0507663	0511525	0515384	0519239	0523091	0526939
13	0530784	0534626	0538464	42299	46131	49959	53783	57605	61423	65237
14	69049	72856	76661	80462	84260	88055	91846	95634	99419	0603200
15	0606978	0610753	0614525	0618293	0622058	0625820	0629578	0633334	0637086	40834
16	44580	48322	52061	55797	59530	63259	66986	70709	74428	78145
17	81859	85569	89276	92980	96681	0700379	0704073	0707765	0711453	0715138
18	0718820	0722499	0726175	0729847	0733517	37184	40847	44507	48164	51819
19	55470	59118	62763	66404	70043	73679	77312	80942	84568	88192
20	91812	95430	99045	0802656	0806265	0809870	0813473	0817073	0820669	0824263
21	0827854	0831441	0835026	38608	42187	45763	49336	52906	56473	60037
22	63598	67157	70712	74265	77814	81361	84905	88446	91984	95519
23	99051	0902581	0906107	0909631	0913152	0916670	0920185	0923697	0927206	0930713
24	0934217	37718	41216	44711	48204	51694	55180	58665	62146	65624
25	69100	72573	76043	79511	82975	86437	89896	93353	96806	1000257
26	1003705	1007151	1010594	1014034	1017471	1020905	1024337	1027766	1031193	1034616
27	38037	41456	44871	48284	51694	55102	58507	61909	65309	68705
28	72100	75491	78880	82267	85650	89031	92410	95785	99159	1102529
29	1105897	1109262	1112625	1115985	1119343	1122698	1126050	1129400	1132747	36092
30	39434	42773	46110	49444	52776	56105	59432	62756	66077	69396
31	72713	76027	79338	82647	85954	89258	92559	95858	99154	1202448
32	1205739	1209028	1212315	1215598	1218880	1222159	1225435	1228709	1231981	35250
33	38516	41781	45042	48301	51558	54813	58065	61314	64561	67806
34	71048	74288	77525	80760	83993	87223	90451	93676	96899	1300119
35	1303338	1306553	1309767	1312978	1316187	1319393	1322597	1325798	1328998	32195
36	35389	38581	41771	44959	48144	51327	54507	57685	60861	64034
37	67206	70375	73541	76705	79877	83027	86184	89339	92492	95643
38	98791	1401937	1405080	1408222	1411361	1414498	1417632	1420765	1423895	1427022
39	1430148	33271	36392	39511	42628	45742	48854	51964	55072	58177
40	61280	64381	67480	70577	73671	76763	79853	82941	86027	89110
41	92191	95270	98347	1501422	1504494	1507564	1510633	1513699	1516762	1519824
42	1522883	1525941	1528996	32049	35100	38149	41195	44240	47282	50322
43	53360	56396	59430	62462	65492	68519	71544	74568	77589	80608
44	83625	86640	89653	92663	95672	98678	1601683	1604685	1607686	1610684
45	1613680	1616674	1619666	1622656	1625644	1628630	31614	34596	37575	40553
46	43529	46502	49474	52443	55411	58376	61340	64301	67261	70218
47	73173	76127	79078	82027	84975	87920	90864	93805	96744	99682
48	1702617	1705551	1708482	1711412	1714339	1717265	1720188	1723110	1726029	1728947
49	31863	34776	37688	40598	43506	46412	49316	52218	55118	58016

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
150	1760913	1763807	1766699	1769590	1772478	1775365	1778250	1781133	1784013	1786892
51	89769	92645	95518	98389	101259	104126	106992	109856	112718	115578
52	1818436	1821292	1824147	1826999	29850	32698	35545	38390	41234	44075
53	46914	49752	52588	55422	58254	61084	63912	66739	69563	72386
54	75207	78026	80844	83659	86473	89285	92095	94903	97710	100514
55	1903317	1906118	1908917	1911715	1914510	1917304	1920096	1922886	1925675	1928461
56	31246	34029	36810	39590	42367	45143	47918	50690	53461	56229
57	58997	61762	64525	67287	70047	72806	75562	78317	81070	83821
58	86571	89319	92065	94809	97552	2000293	2003032	2005769	2008505	2011239
59	2013971	2016702	2019431	2022158	2024883	27607	30329	33049	35768	38485
60	41200	43913	46625	49335	52044	54750	57455	60159	62860	65560
61	68259	70955	73650	76344	79035	81725	84414	87100	89785	92468
62	95150	97830	2100508	2103185	2105860	2108534	2111205	2113876	2116544	2119211
63	2121876	2124540	27202	29862	32521	35178	37833	40487	43139	45790
64	48438	51086	53732	56376	59018	61659	64298	66936	69572	72207
65	74839	77471	80100	82729	85355	87980	90603	93225	95845	98464
66	2201081	2203696	2206310	2208922	2211533	2214142	2216750	2219356	2221960	2224563
67	27165	29764	32363	34959	37555	40148	42740	45331	47920	50507
68	53093	55677	58260	60841	63421	65999	68576	71151	73724	76296
69	78867	81436	84004	86570	89134	91697	94258	96818	99377	2301934
70	2304489	2307043	2309596	2312146	2314696	2317244	2319790	2322335	2324879	27421
71	29961	32500	35038	37574	40108	42641	45173	47703	50232	52759
72	55284	57809	60331	62853	65373	67891	70408	72923	75437	77950
73	80461	82971	85479	87986	90491	92995	95497	97998	2400498	2402996
74	2405492	2407988	2410482	2412974	2415465	2417954	2420442	2422929	25414	27898
75	30380	32861	35341	37819	40296	42771	45245	47718	50189	52658
76	55127	57594	60059	62523	64986	67447	69907	72365	74823	77278
77	79733	82186	84637	87087	89536	91984	94430	96874	99318	2501759
78	2504200	2506639	2509077	2511513	2513949	2516382	2518815	2521246	2523675	26103
79	28530	30956	33380	35803	38224	40645	43063	45481	47897	50312
80	52725	55137	57548	59957	62365	64772	67177	69582	71984	74386
81	76786	79185	81582	83978	86373	88766	91158	93549	95939	98327
82	2600714	2603099	2605484	2607867	2610248	2612629	2615008	2617385	2619762	2622137
83	24511	26883	29255	31625	33993	36361	38727	41092	43455	45817
84	48178	50538	52896	55253	57609	59964	62317	64669	67020	69369
85	71717	74064	76410	78754	81097	83439	85780	88119	90457	92794
86	95129	97464	99797	2702029	2704459	2706888	2709316	2711743	2714169	2716593
87	2718416	2720838	2723258	2725678	2728096	2730513	2732928	2735343	2737756	2740168
88	41578	43888	46196	48503	50809	53114	55417	57719	60020	62320
89	64618	66915	69211	71506	73800	76092	78383	80673	82962	85250
90	87536	89821	92105	94388	96669	98950	2801229	2803507	2805784	2808059
91	2810334	2812607	2814879	2817150	2819419	2821688	23955	26221	28486	30750
92	33012	35274	37534	39793	42051	44307	46563	48817	51070	53322
93	55573	57823	60071	62319	64565	66810	69054	71296	73538	75778
94	78017	80255	82492	84728	86963	89196	91428	93660	95890	98118
95	2900346	2902573	2904798	2907022	2909246	2911468	2913689	2915908	2918127	2920344
96	22561	24776	26990	29203	31415	33626	35835	38044	40251	42457
97	44662	46866	49069	51271	53471	55671	57869	60067	62263	64458
98	66652	68845	71037	73227	75417	77605	79792	81979	84164	86348
99	88531	90713	92893	95073	97252	99429	3001605	3003781	3005955	3008128

SEVEN-FIGURE LOGARITHMS

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200	3010300	3012471	3014641	3016809	3018977	3021144	3023309	3025474	3027637	3029799
1	31961	34121	36280	38438	40595	42751	44905	47059	49212	51363
2	53514	55663	57812	59959	62105	64250	66394	68537	70680	72820
3	74960	77099	79237	81374	83509	85644	87778	89910	92042	94172
4	96302	98430	3100557	3102684	3104809	3106933	3109056	3111178	3113300	3115420
5	3117539	3119657	21774	23889	26004	28118	30231	32343	34454	36563
6	38672	40780	42887	44992	47097	49201	51303	53405	55505	57605
7	59703	61801	63898	65993	68088	70181	72273	74365	76455	78545
8	80633	82721	84807	86893	88977	91061	93143	95224	97305	99384
9	3201463	3203540	3205617	3207692	3209767	3211840	3213913	3215984	3218055	3220124
10	22193	24261	26327	28393	30457	32521	34584	36645	38706	40766
11	42825	44882	46939	48995	51050	53104	55157	57209	59260	61310
12	63359	65407	67454	69500	71545	73589	75633	77675	79716	81757
13	83796	85834	87872	89909	91944	93979	96012	98045	3300077	3302108
14	3304138	3306167	3308195	3310222	3312248	3314273	3316297	3318320	20343	22364
15	24385	26404	28423	30440	32457	34473	36488	38501	40514	42526
16	44538	46548	48557	50565	52573	54579	56585	58589	60593	62596
17	64597	66598	68598	70597	72595	74593	76589	78584	80579	82572
18	84565	86557	88547	90537	92526	94514	96502	98488	3400473	3402458
19	3404441	3406424	3408405	3410386	3412366	3414345	3416323	3418301	20277	22252
20	24227	26200	28173	30145	32116	34086	36055	38023	39991	41957
21	43923	45887	47851	49814	51776	53737	55698	57657	59615	61573
22	63530	65486	67441	69395	71348	73300	75252	77202	79152	81101
23	83049	84996	86942	88887	90832	92775	94718	96660	98601	3500541
24	3502480	3504419	3506356	3508293	3510229	3512163	3514098	3516031	3517963	19895
25	21825	23755	25684	27612	29539	31465	33391	35316	37239	39162
26	41084	43006	44926	46846	48764	50682	52599	54515	56431	58345
27	60259	62171	64083	65994	67905	69814	71723	73630	75537	77443
28	79348	81253	83156	85059	86961	88862	90762	92662	94560	96458
29	98355	3600251	3602146	3604041	3605934	3607827	3609719	3611610	3613500	3615390
30	3617278	19166	21053	22939	24825	26709	28593	30476	32358	34239
31	36120	37999	39878	41756	43634	45510	47386	49260	51134	53007
32	54880	56751	58622	60492	62361	64230	66097	67964	69830	71695
33	73559	75423	77285	79147	81009	82866	84728	86587	88445	90302
34	92159	94014	95869	97723	99576	3701428	3703280	3705131	3706981	3708830
35	3710679	3712526	3714373	3716219	3718065	19909	21753	23596	25438	27279
36	29120	30960	32799	34637	36475	38311	40147	41983	43817	45651
37	47483	49316	51147	52977	54807	56636	58464	60292	62119	63944
38	65770	67594	69418	71240	73063	74884	76704	78524	80343	82161
39	83979	85796	87612	89427	91241	93055	94868	96680	98492	3800302
40	3802112	3803922	3805730	3807538	3809345	3811151	3812956	3814761	3816565	18368
41	20170	21972	23773	25573	27373	29171	30969	32767	34563	36359
42	38154	39948	41741	43534	45326	47117	48908	50698	52487	54275
43	56063	57850	59636	61421	63206	64990	66773	68555	70337	72118
44	73808	75678	77457	79235	81012	82789	84565	86340	88114	89888
45	91661	93433	95205	96975	98746	3900515	3902284	3904052	3905810	3907585
46	3909351	3911116	3912880	3914644	3916407	18169	19931	21691	23452	25211
47	26970	28727	30485	32241	33997	35752	37506	39260	41013	42765
48	44517	46268	48018	49767	51516	53264	55011	56758	58504	60249
49	61993	63737	65480	67223	68964	70705	72446	74185	75924	77663

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250	3979400	3981137	3982873	3984608	3986343	3988077	3989811	3991543	3993275	3995007
51	96737	98467	4000196	4001925	4003653	4005380	4007106	4008832	4010557	4012282
52	4014005	4015728	17451	19173	20894	22614	24333	26052	27771	29488
53	31205	32921	34637	36352	38066	39780	41492	43205	44916	46627
54	48337	50047	51755	53464	55171	56878	58584	60289	61994	63698
55	65402	67105	68807	70508	72209	73909	75608	77307	79005	80703
56	82400	84096	85791	87486	89180	90874	92567	94259	95950	97641
57	99331	4101021	4102710	4104398	4106085	4107772	4109459	4111144	4112829	4114513
58	4116197	17880	19562	21244	22925	24605	26285	27964	29643	31321
59	32998	34674	36350	38025	39700	41374	43047	44719	46391	48063
60	49733	51404	53073	54742	56410	58077	59744	61410	63076	64741
61	66405	68069	69732	71394	73056	74717	76377	78037	79696	81355
62	83013	84670	86327	87983	89638	91293	92947	94601	96254	97906
63	99557	4201208	4202859	4204509	4206158	4207806	4209454	4211101	4212748	4214394
64	16039	17684	19328	20972	22615	24257	25898	27539	29180	30820
65	32459	34097	35735	37372	39009	40645	42281	43916	45550	47183
66	48816	50449	52081	53712	55342	56972	58601	60230	61858	63486
67	65113	66739	68365	69990	71614	73238	74861	76484	78106	79727
68	81348	82968	84588	86207	87825	89443	91060	92677	94293	95908
69	97523	99137	4300751	4302364	4303976	4305588	4307199	4308809	4310419	4312029
70	4313638	4315246	16853	18460	20067	21673	23278	24883	26487	28090
71	29693	31295	32897	34498	36098	37698	39298	40896	42495	44092
72	45689	47285	48881	50476	52071	53665	55259	56851	58444	60035
73	61626	63217	64807	66396	67985	69573	71161	72748	74334	75920
74	77506	79090	80675	82258	83841	85423	87005	88587	90167	91747
75	93327	94906	96484	98062	99639	4401216	4402792	4404368	4405943	4407517
76	4409091	4410664	4412237	4413809	4415380	16951	18522	20092	21661	23230
77	24798	26365	27932	29499	31065	32630	34195	35759	37322	38885
78	40448	42010	43571	45132	46692	48252	49811	51370	52928	54485
79	56042	57598	59154	60709	62264	63818	65372	66925	68477	70029
80	71580	73131	74681	76231	77780	79329	80877	82424	83971	85517
81	87063	88608	90153	91697	93241	94784	96327	97868	99410	4500951
82	4502491	4504031	4505570	4507109	4508647	4510185	4511722	4513258	4514794	10329
83	17864	19399	20932	22466	23998	25531	27062	28593	30124	31654
84	33183	34712	36241	37769	39296	40823	42349	43875	45400	46924
85	48449	49972	51495	53018	54540	56061	57582	59102	60622	62142
86	63660	65179	66696	68213	69730	71246	72762	74277	75791	77305
87	78819	80332	81844	83356	84868	86378	87889	89399	90908	92417
88	93925	95433	96940	98446	99953	4601458	4602963	4604468	4605972	4607475
89	4608978	4610481	4611983	4613484	4614985	16486	17986	19485	20984	22482
90	23980	25477	26974	28470	29966	31461	32956	34450	35944	37437
91	38930	40422	41914	43405	44895	46386	47875	49364	50853	52341
92	53829	55316	56802	58288	59774	61259	62743	64227	65711	67194
93	68676	70158	71640	73121	74601	76081	77561	79039	80518	81996
94	83473	84950	86427	87903	89378	90853	92327	93801	95275	96748
95	98220	99692	4701164	4702634	4704105	4705575	4707044	4708513	4709982	4711450
96	4712917	4714384	15851	17317	18782	20247	21711	23175	24639	26102
97	27564	29027	30488	31949	33410	34870	36329	37788	39247	40705
98	42163	43620	45076	46533	47988	49443	50898	52352	53806	55259
99	56712	58164	59616	61067	62518	63968	65418	66867	68316	69765

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300	4771213	4772660	4774107	4775553	4776999	4778445	4779890	4781334	4782778	4784222
1	85665	87108	88550	89991	91432	92873	94313	95753	97192	98631
2	4800069	4801507	4802945	4804381	4805818	4807254	4808689	4810124	4811559	4812993
3	14426	15859	17292	18724	20156	21587	23018	24448	25878	27307
4	28736	30164	31592	33020	34446	35873	37299	38725	40150	41574
5	42998	44422	45845	47268	48690	50112	51533	52954	54375	55795
6	57214	58633	60052	61470	62888	64305	65722	67138	68554	69969
7	71384	72798	74212	75626	77039	78451	79863	81275	82686	84097
8	85507	86917	88326	89735	91144	92552	93959	95366	96773	98179
9	99585	4900900	4902395	4903799	4905203	4906607	4908010	4909412	4910814	4912216
10	4913617	15018	16418	17818	19217	20616	22015	23413	24810	26207
11	27604	29000	30396	31791	33186	34581	35974	37368	38761	40154
12	41546	42938	44329	45720	47110	48500	49890	51279	52667	54056
13	55443	56831	58218	59604	60990	62375	63761	65145	66529	67913
14	69296	70679	72062	73444	74825	76206	77587	78967	80347	81727
15	83106	84484	85862	87240	88617	89994	91370	92746	94121	95496
16	96871	98245	99619	5000992	5002365	5003737	5005109	5006481	5007852	5009222
17	5010593	5011962	5013332	14701	16069	17437	18805	20172	21539	22905
18	24271	25637	27002	28366	29731	31094	32458	33821	35183	36545
19	37907	39268	40629	41989	43349	44709	46068	47426	48785	50142
20	51500	52857	54213	55569	56925	58280	59635	60990	62344	63697
21	65050	66403	67755	69107	70459	71810	73160	74511	75860	77210
22	78559	79907	81255	82603	83950	85297	86644	87990	89335	90680
23	92025	93370	94714	96057	97400	98743	5100085	5101427	5102768	5104109
24	5105450	5106790	5108130	5109469	5110808	5112147	13485	14823	16160	17497
25	18834	20170	21505	22841	24175	25510	26844	28178	29511	30844
26	32176	33508	34840	36171	37502	38832	40162	41491	42820	44149
27	45478	46805	48133	49460	50787	52113	53439	54764	56089	57414
28	58738	60062	61386	62709	64031	65354	66676	67997	69318	70639
29	71959	73279	74598	75917	77236	78554	79872	81189	82507	83823
30	85139	86455	87771	89086	90400	91715	93028	94342	95655	96968
31	98280	99592	5200903	5202214	5203525	5204835	5206145	5207455	5208764	5210073
32	5211381	5212689	13996	15303	16610	17916	19222	20528	21833	23138
33	24442	25746	27050	28353	29656	30958	32260	33562	34863	36164
34	37465	38765	40064	41364	42663	43961	45259	46557	47854	49151
35	50448	51744	53040	54336	55631	56925	58220	59513	60807	62100
36	63393	64685	65977	67269	68560	69851	71141	72431	73721	75010
37	76209	77588	78876	80163	81451	82738	84024	85311	86598	87882
38	89167	90452	91736	93020	94304	95587	96870	98152	99434	5300716
39	5301907	5303278	5304658	5306039	5307418	5308798	5309677	5310955	5312234	13512
40	14789	16066	17343	18619	19896	21171	22446	23721	24996	26270
41	27544	28817	30090	31363	32635	33907	35179	36450	37721	38991
42	40261	41531	42800	44069	45338	46606	47874	49141	50408	51675
43	52041	54207	55473	56738	58003	59267	60532	61795	63059	64322
44	65584	66847	68109	69370	70631	71892	73153	74413	75673	76932
45	78101	79450	80708	81966	83223	84481	85737	86994	88250	89506
46	90761	92016	93271	94525	95779	97032	98286	99538	5400791	5402043
47	5403295	5404546	5405797	5407048	5408298	5409548	5410798	5412047	13296	14544
48	15792	17040	18288	19535	20781	22028	23274	24519	25765	27010
49	28254	29498	30742	31986	33229	34472	35714	36956	38198	39439

SEVEN-FIGURE LOGARITHMS

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350	5440680	5441921	5443161	5444401	5445641	5446880	5448119	5449358	5450596	5451834
51	53071	54308	55545	50781	58018	59253	60489	61724	62958	64193
52	65427	66660	67894	69126	70359	71591	72823	74055	75286	76517
53	77747	78977	80207	81436	82665	83894	85123	86351	87578	88806
54	90033	91259	92486	93712	94937	96162	97387	98612	99836	5501060
55	5502284	5503507	5504730	5505952	5507174	5508396	5509618	5510839	5512059	13280
56	14500	15720	16939	18158	19377	20595	21813	23031	24248	25465
57	26682	27899	29115	30330	31545	32760	33975	35189	36403	37617
58	38830	40043	41256	42468	43680	44892	46103	47314	48524	49735
59	50944	52154	53363	54572	55781	56989	58197	59404	60612	61818
60	63025	64231	65437	66643	67848	69053	70257	71461	72665	73869
61	75072	76275	77477	78680	79881	81083	82284	83485	84686	85886
62	87086	88285	89484	90683	91882	93080	94278	95476	96673	97870
63	99066	5600262	5601458	5602654	5603849	5605044	5606239	5607433	5608627	5609821
64	5611014	12207	13399	14592	15784	16975	18167	19358	20548	21739
65	22929	24118	25308	26497	27685	28874	30062	31250	32437	33624
66	34811	35997	37183	38369	39555	40740	41925	43109	44293	45477
67	46661	47844	49027	50209	51392	52573	53755	54936	56117	57298
68	58478	59658	60838	62017	63196	64375	65553	66731	67909	69087
69	70264	71440	72617	73793	74969	76144	77320	78495	79669	80843
70	82017	83191	84364	85537	86710	87882	89054	90226	91397	92568
71	93739	94910	96080	97249	98419	99588	5700757	5701926	5703094	5704262
72	5705429	5706597	5707764	5708930	5710097	5711263	12429	13594	14759	15924
73	17088	18252	19416	20580	21743	22906	24069	25231	26393	27555
74	28716	29877	31038	32198	33358	34518	35678	36837	37996	39154
75	40313	41471	42628	43786	44943	46099	47256	48412	49568	50723
76	51878	53033	54188	55342	56496	57650	58803	59956	61109	62261
77	63414	64565	65717	66868	68019	69170	70320	71470	72620	73769
78	74918	76067	77215	78363	79511	80659	81806	82953	84100	85246
79	86392	87538	88683	89828	90973	92118	93262	94406	95550	96693
80	97836	98979	5800121	5801263	5802405	5803547	5804688	5805829	5806969	5808110
81	5809250	5810389	11529	12668	13807	14945	16084	17222	18359	19497
82	20634	21770	22907	24043	25179	26314	27450	28585	29719	30854
83	31988	33122	34255	35388	36521	37654	38786	39918	41050	42181
84	43312	44443	45574	46704	47834	48963	50093	51222	52351	53479
85	54607	55735	56863	57990	59117	60244	61370	62496	63622	64748
86	65873	66998	68123	69247	70371	71495	72618	73742	74865	75987
87	77110	78232	79353	80475	81596	82717	83838	84958	86078	87198
88	88317	89436	90555	91674	92792	93910	95028	96145	97263	98379
89	99496	5900612	5901728	5902844	5903959	5905075	5906189	5907304	5908418	5909532
90	5910646	11760	12873	13986	15098	16210	17322	18434	19546	20657
91	21768	22878	23988	25098	26208	27318	28427	29536	30644	31753
92	32861	33968	35076	36183	37290	38397	39503	40609	41715	42820
93	43926	45030	46135	47239	48344	49447	50551	51654	52757	53860
94	54962	56064	57166	58268	59369	60470	61571	62671	63771	64871
95	65971	67070	68169	69268	70367	71465	72563	73661	74758	75855
96	76952	78048	79145	80241	81336	82432	83527	84622	85717	86811
97	87905	88999	90092	91186	92279	93371	94464	95556	96648	97739
98	98831	99922	6001013	6002103	6003193	6004283	6005373	6006462	6007551	6008640
99	6009729	6010817	11905	12993	14081	15168	16255	17341	18428	19514

SEVEN-FIGURE LOGARITHMS

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400	6020600	6021686	6022771	6023856	6024941	6026025	6027109	6028193	6029277	6030361
1	31444	32527	33609	34692	35774	36855	37937	39018	40099	41180
2	42261	43341	44421	45500	46580	47659	48738	49816	50895	51973
3	53050	54128	55205	56282	57359	58435	59512	60587	61663	62739
4	63814	64889	65963	67037	68111	69185	70259	71332	72405	73478
5	74550	75622	76694	77766	78837	79909	80979	82050	83120	84191
6	85260	86330	87399	88468	89537	90605	91674	92742	93809	94877
7	95944	97011	98078	99144	6100210	6101276	6102342	6103407	6104472	6105537
8	6106602	6107666	6108730	6109794	10857	11921	12984	14046	15109	16171
9	17233	18295	19356	20417	21478	22539	23599	24660	25720	26779
10	27839	28898	29957	31015	32074	33132	34189	35247	36304	37361
11	38418	39475	40531	41587	42643	43698	44754	45809	46863	47918
12	48972	50026	51080	52133	53187	54240	55292	56345	57397	58449
13	59501	60552	61603	62654	63705	64755	65805	66855	67905	68954
14	70003	71052	72101	73149	74197	75245	76293	77340	78387	79434
15	80481	81527	82573	83619	84665	85710	86755	87800	88845	89889
16	90933	91977	93021	94064	95107	96150	97193	98235	99277	6200319
17	6201361	6202402	6203443	6204484	6205524	6206565	6207605	6208645	6209684	10724
18	11763	12802	13840	14879	15917	16955	17992	19030	20067	21104
19	22140	23177	24213	25249	26284	27320	28355	29390	30424	31459
20	32493	33527	34560	35594	36627	37660	38693	39725	40757	41789
21	42821	43852	44884	45915	46945	47976	49006	50036	51066	52095
22	53125	54154	55182	56211	57239	58267	59295	60322	61350	62377
23	63404	64430	65457	66483	67509	68534	69560	70585	71610	72634
24	73659	74683	75707	76730	77754	78777	79800	80823	81845	82867
25	83889	84911	85933	86954	87975	88996	90016	91037	92057	93076
26	94096	95115	96134	97153	98172	99190	6300209	6301226	6302244	6303262
27	6304279	6305296	6306312	6307329	6308345	6309361	10377	11393	12408	13423
28	14438	15452	16467	17481	18495	19508	20522	21535	22548	23560
29	24573	25585	26597	27609	28620	29632	30643	31654	32664	33674
30	34685	35694	36704	37713	38723	39732	40740	41749	42757	43765
31	44773	45780	46788	47795	48801	49808	50814	51820	52826	53832
32	54837	55843	56848	57852	58857	59861	60865	61869	62873	63876
33	64879	65882	66884	67887	68889	69891	70893	71894	72895	73897
34	74897	75898	76898	77898	78898	79898	80897	81898	82895	83894
35	84893	85891	86889	87887	88884	89882	90879	91876	92872	93869
36	94805	95801	96857	97852	98847	99842	6400837	6401832	6402826	6403820
37	6404814	6405808	6406802	6407795	6408788	6409781	10773	11765	12758	13749
38	14741	15733	16724	17715	18705	19696	20686	21676	22666	23656
39	24645	25634	26623	27612	28601	29589	30577	31565	32552	33540
40	34527	35514	36500	37487	38473	39459	40445	41431	42416	43401
41	44386	45371	46355	47339	48323	49307	50291	51274	52257	53240
42	54223	55205	56187	57169	58151	59133	60114	61095	62076	63057
43	64037	65018	65998	66977	67957	68936	69915	70894	71873	72851
44	73830	74808	75786	76763	77741	78718	79695	80671	81648	82624
45	83600	84576	85552	86527	87502	88477	89452	90426	91401	92375
46	93349	94322	95296	96269	97242	98215	99187	6500160	6501132	6502104
47	6503075	6504047	6505018	6505989	6506960	6507930	6508901	09871	10841	11811
48	12780	13749	14719	15687	16656	17624	18593	19561	20528	21496
49	22463	23431	24397	25364	26331	27297	28263	29229	30195	31160

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	0	1	2	3	4	5	6	7	8	9
450	6532125	6533090	6534055	6535019	6535984	6536948	6537912	6538876	6539839	6540802
51	41765	42728	43691	44653	45616	46578	47539	48501	49462	50423
52	51384	52345	53306	54266	55226	56186	57145	58105	59064	60023
53	60982	61941	62899	63857	64815	65773	66730	67688	68645	69602
54	70559	71515	72471	73427	74383	75339	76294	77250	78205	79159
55	80114	81068	82023	82977	83930	84884	85837	86790	87743	88696
56	89648	90601	91553	92505	93456	94408	95359	96310	97261	98212
57	99162	6600112	6601062	6602012	6602962	6603911	6604860	6605809	6606758	6607706
58	6608655	09603	10551	11499	12446	13393	14341	15287	16234	17181
59	18127	19073	20019	20964	21910	22855	23800	24745	25690	26634
60	27578	28522	29466	30410	31353	32296	33239	34182	35125	36067
61	37009	37951	38893	39835	40776	41717	42658	43599	44539	45480
62	46420	47360	48299	49239	50178	51117	52056	52995	53934	54872
63	55810	56748	57686	58623	59560	60497	61434	62371	63307	64244
64	65180	66116	67051	67987	68922	69857	70792	71727	72661	73595
65	74530	75463	76397	77331	78264	79197	80130	81062	81995	82927
66	83859	84791	85723	86654	87585	88516	89447	90378	91308	92239
67	93169	94099	95028	95958	96887	97816	98745	99674	6700602	6701530
68	6702459	6703386	6704314	6705242	6706169	6707096	6708023	6708950	09876	10802
69	11728	12654	13580	14506	15431	16356	17281	18206	19130	20054
70	20979	21903	22826	23750	24673	25596	26519	27442	28365	29287
71	30209	31131	32053	32974	33896	34817	35738	36659	37579	38500
72	39420	40340	41260	42179	43099	44018	44937	45856	46775	47693
73	48611	49529	50447	51365	52283	53200	54117	55034	55951	56867
74	57783	58700	59615	60531	61447	62362	63277	64192	65107	66022
75	66936	67850	68764	69678	70592	71505	72418	73332	74244	75157
76	76070	76982	77894	78806	79718	80629	81540	82452	83362	84273
77	85184	86094	87004	87914	88824	89734	90643	91552	92461	93370
78	94279	95187	96096	97004	97912	98819	99727	6800634	6801541	6802448
79	6803355	6804262	6805168	6806074	6806980	6807886	6808792	09697	10602	11507
80	12412	13317	14222	15126	16030	16934	17838	18741	19645	20548
81	21451	22354	23256	24159	25061	25963	26865	27766	28668	29569
82	30470	31371	32272	33173	34073	34973	35873	36773	37673	38572
83	39471	40370	41269	42168	43066	43965	44863	45761	46659	47556
84	48454	49351	50248	51145	52041	52938	53834	54730	55626	56522
85	57417	58313	59208	60103	60998	61892	62787	63681	64575	65469
86	66363	67256	68150	69043	69936	70828	71721	72613	73506	74398
87	75290	76181	77073	77964	78855	79746	80637	81528	82418	83308
88	84198	85088	85978	86867	87757	88646	89535	90423	91312	92200
89	93089	93977	94864	95752	96640	97527	98414	99301	6900188	6901074
90	6901961	6902847	6903733	6904619	6905505	6906390	6907275	6908161	09046	09930
91	10815	11699	12584	13468	14352	15235	16119	17002	17885	18768
92	19651	20534	21416	22298	23180	24062	24944	25826	26707	27588
93	28469	29350	30231	31111	31991	32872	33752	34631	35511	36390
94	37269	38149	39027	39906	40785	41663	42541	43419	44297	45175
95	46052	46929	47806	48683	49560	50437	51313	52189	53065	53941
96	54817	55692	56568	57443	58318	59193	60067	60942	61816	62690
97	63564	64438	65311	66185	67058	67931	68804	69676	70549	71421
98	72293	73165	74037	74909	75780	76652	77523	78394	79264	80135
99	81005	81876	82746	83616	84485	85355	86224	87093	87963	88831

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500	6989700	6990569	6991437	6992305	6993173	6994041	6994908	6995776	6996643	6997510
1	98377	99244	7000111	7000977	7001843	7002709	7003575	7004441	7005307	7006172
2	7007037	7007902	08767	09632	10496	11361	12225	13089	13953	14816
3	15680	16543	17406	18269	19132	19995	20857	21720	22582	23444
4	24305	25167	26028	26890	27751	28612	29472	30333	31193	32055
5	32914	33774	34633	35498	36352	37212	38071	38930	39788	40647
6	41505	42363	43221	44079	44937	45794	46652	47509	48366	49223
7	50080	50936	51792	52649	53505	54360	55216	56072	56927	57782
8	58637	59492	60347	61201	62055	62910	63764	64617	65471	66325
9	67178	68031	68884	69737	70589	71442	72294	73146	73998	74850
10	75702	76553	77405	78256	79107	79957	80808	81659	82509	83359
11	84209	85059	85908	86758	87607	88456	89305	90154	91003	91851
12	92700	93548	94396	95244	96091	96939	97786	98633	99480	7100327
13	7101174	7102020	7102866	7103713	7104559	7105404	7106250	7107096	7107941	08786
14	09631	10476	11321	12165	13010	13854	14698	15542	16385	17229
15	18072	18915	19759	20601	21444	22287	23129	23971	24813	25655
16	26497	27339	28180	29021	29862	30703	31544	32385	33225	34065
17	34905	35745	36585	37425	38264	39104	39943	40782	41620	42459
18	43298	44136	44974	45812	46650	47488	48325	49162	50000	50837
19	51674	52510	53347	54183	55019	55856	56691	57527	58363	59198
20	60033	60869	61703	62538	63373	64207	65042	65876	66710	67544
21	68377	69211	70044	70877	71710	72543	73376	74208	75041	75873
22	76705	77537	78369	79200	80032	80863	81694	82525	83356	84186
23	85017	85847	86677	87507	88337	89167	89996	90826	91655	92484
24	93313	94142	94970	95799	96627	97455	98283	99111	99938	7200766
25	7201593	7202420	7203247	7204074	7204901	7205727	7206554	7207380	7208206	09032
26	09857	10683	11508	12334	13159	13984	14809	15633	16458	17282
27	18106	18930	19754	20578	21401	22225	23048	23871	24694	25517
28	26339	27162	27984	28806	29628	30450	31272	32093	32914	33736
29	34557	35378	36198	37019	37839	38660	39480	40300	41120	41939
30	42759	43578	44397	45216	46035	46854	47672	48491	49309	50127
31	50945	51763	52581	53398	54216	55033	55850	56667	57483	58300
32	59116	59933	60749	61565	62380	63196	64012	64827	65642	66457
33	67272	68087	68901	69716	70530	71344	72158	72972	73786	74599
34	75413	76226	77039	77852	78664	79477	80290	81102	81914	82726
35	83538	84350	85161	85972	86784	87595	88406	89216	90027	90838
36	91648	92458	93268	94078	94888	95697	96507	97316	98125	98934
37	99743	7300552	7301360	7302168	7302977	7303785	7304593	7305400	7306208	7307015
38	7307823	08630	09437	10244	11051	11857	12663	13470	14276	15082
39	15888	16693	17499	18304	19109	19914	20719	21524	22329	23133
40	23938	24742	25546	26350	27153	27957	28760	29564	30367	31170
41	31973	32775	33578	34380	35183	35985	36787	37588	38390	39192
42	39993	40794	41595	42396	43197	43997	44798	45598	46398	47198
43	47998	48798	49598	50397	51196	51995	52794	53593	54392	55191
44	55989	56787	57585	58383	59181	59979	60776	61574	62371	63168
45	63965	64762	65558	66355	67151	67948	68744	69540	70335	71131
46	71926	72722	73517	74312	75107	75902	76696	77491	78285	79079
47	79873	80667	81461	82254	83048	83841	84634	85427	86220	87013
48	87806	88598	89390	90182	90974	91766	92558	93350	94141	94932
49	95723	96514	97305	98096	98887	99677	7400467	7401257	7402047	7402837

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550	7403627	7404416	7405206	7405995	7406784	7407573	7408362	7409151	7409939	7410728
51	11516	12304	13092	13880	14668	15455	16243	17030	17817	18604
52	19391	20177	20964	21750	22537	23323	24109	24895	25680	26466
53	27251	28037	28822	29607	30392	31176	31961	32745	33530	34314
54	35098	35882	36665	37449	38232	39016	39799	40582	41365	42147
55	42930	43712	44495	45277	46059	46841	47622	48404	49185	49967
56	50748	51529	52310	53091	53871	54652	55432	56212	56992	57772
57	58552	59332	60111	60890	61670	62449	63228	64006	64785	65564
58	66342	67120	67898	68676	69454	70232	71009	71787	72564	73341
59	74118	74895	75672	76448	77225	78001	78777	79553	80329	81105
60	81880	82656	83431	84206	84981	85756	86531	87306	88080	88854
61	89629	90403	91177	91950	92724	93498	94271	95044	95817	96590
62	97363	98136	98908	99681	7500453	7501225	7501997	7502769	7503541	7504312
63	7505084	7505855	7506626	7507398	08168	08939	09710	10480	11251	12021
64	12791	13561	14331	15101	15870	16639	17409	18178	18947	19716
65	20484	21253	22022	22790	23558	24326	25094	25862	26629	27397
66	28164	28932	29699	30466	31232	31999	32766	33532	34298	35065
67	35831	36596	37362	38128	38893	39659	40424	41189	41954	42719
68	43483	44248	45012	45777	46541	47305	48069	48832	49596	50359
69	51123	51886	52649	53412	54175	54937	55700	56462	57224	57987
70	58749	59510	60272	61034	61795	62556	63318	64079	64840	65600
71	66361	67122	67882	68642	69402	70162	70922	71682	72442	73201
72	73960	74719	75479	76237	76996	77755	78513	79272	80030	80788
73	81546	82304	83062	83819	84577	85334	86091	86848	87605	88362
74	89119	89875	90632	91388	92144	92900	93656	94412	95168	95923
75	96678	97434	98189	98944	99699	7600453	7601208	7601962	7602717	7603471
76	7604225	7604979	7605733	7606486	7607240	07993	08746	09500	10253	11005
77	11758	12511	13263	14016	14768	15520	16272	17024	17775	18527
78	19278	20030	20781	21532	22283	23034	23784	24535	25285	26035
79	26786	27536	28286	29035	29785	30534	31284	32033	32782	33531
80	34280	35029	35777	36526	37274	38022	38770	39518	40266	41014
81	41761	42509	43256	44003	44750	45497	46244	46991	47737	48484
82	49230	49976	50722	51468	52214	52959	53705	54450	55195	55941
83	56686	57430	58175	58920	59664	60409	61153	61897	62641	63385
84	64128	64872	65616	66359	67102	67845	68588	69331	70074	70816
85	71559	72301	73043	73785	74527	75269	76011	76752	77494	78235
86	78976	79717	80458	81199	81940	82680	83421	84161	84901	85641
87	86381	87121	87860	88600	89339	90079	90818	91557	92296	93035
88	93773	94512	95250	95988	96727	97465	98203	98940	99678	7700416
89	7701153	7701890	7702627	7703364	7704101	7704838	7705575	7706311	7707048	07784
90	08520	09256	09992	10728	11463	12199	12934	13670	14405	15140
91	15875	16610	17344	18079	18813	19547	20282	21016	21750	22483
92	23217	23951	24684	25417	26150	26884	27616	28349	29082	29815
93	30547	31279	32011	32743	33475	34207	34939	35670	36402	37133
94	37864	38596	39326	40057	40788	41519	42249	42979	43710	44440
95	45170	45900	46629	47359	48088	48818	49547	50276	51005	51734
96	52463	53191	53920	54648	55376	56104	56832	57560	58288	59016
97	59743	60471	61198	61925	62652	63379	64106	64833	65559	66286
98	67012	67738	68464	69190	69916	70642	71367	72093	72818	73543
99	74268	74993	75718	76443	77167	77892	78616	79340	80065	80789

SEVEN-FIGURE LOGARITHMS

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600	7781513	7782236	7782960	7783683	7784407	7785130	7785853	7786576	7787299	7788022
1	88745	89467	90190	90912	91634	92356	93078	93800	94522	95243
2	95965	96686	97408	98129	98850	99571	8000291	7801012	7801732	7802453
3	7803173	7803893	7804613	7805333	7806053	7806773	07492	08212	08931	09650
4	10369	11088	11807	12526	13245	13963	14681	15400	16118	16836
5	17554	18272	18989	19707	20424	21141	21859	22576	23293	24010
6	24726	25443	26159	26876	27592	28308	29024	29740	30456	31171
7	31887	32602	33318	34033	34748	35463	36178	36892	37607	38321
8	39036	39750	40464	41178	41892	42606	43319	44033	44746	45460
9	46173	46886	47599	48312	49024	49737	50450	51162	51874	52586
10	53298	54010	54722	55434	56145	56857	57568	58279	58990	59701
11	60412	61123	61833	62544	63254	63965	64675	65385	66095	66805
12	67514	68224	68933	69643	70352	71061	71770	72479	73188	73896
13	74605	75313	76021	76730	77438	78146	78854	79561	80269	80976
14	81684	82391	83098	83805	84512	85219	85926	86632	87339	88045
15	88751	89457	90163	90869	91575	92281	92986	93692	94397	95102
16	95807	96512	97217	97922	98626	99331	7900035	7900739	7901444	7902148
17	7902852	7903555	7904259	7904963	7905666	7906370	07073	07776	08479	09182
18	09885	10587	11290	11992	12695	13397	14099	14801	15503	16205
19	16906	17608	18309	19011	19712	20413	21114	21815	22516	23216
20	23917	24617	25318	26018	26718	27418	28118	28817	29517	30217
21	30916	31615	32314	33014	33712	34411	35110	35809	36507	37206
22	37904	38602	39300	39998	40696	41394	42091	42789	43486	44183
23	44880	45578	46274	46971	47668	48365	49061	49757	50454	51150
24	51846	52542	53238	53933	54629	55324	56020	56715	57410	58105
25	58800	59495	60190	60884	61579	62273	62967	63662	64356	65050
26	65743	66437	67131	67824	68517	69211	69904	70597	71290	71983
27	72675	73368	74060	74753	75445	76137	76829	77521	78213	78905
28	79596	80288	80979	81671	82362	83053	83744	84435	85125	85816
29	86506	87197	87887	88577	89267	89957	90647	91337	92027	92716
30	93405	94095	94784	95473	96162	96851	97540	98228	98917	99605
31	8000294	8000982	8001670	8002358	8003046	8003734	8004421	8005109	8005796	8006484
32	07171	07858	08545	09232	09919	10605	11292	11978	12665	13351
33	14037	14723	15409	16095	16781	17466	18152	18837	19522	20208
34	20893	21578	22262	22947	23632	24316	25001	25685	26369	27053
35	27737	28421	29105	29789	30472	31156	31839	32522	33205	33888
36	34571	35254	35937	36619	37302	37984	38666	39348	40031	40712
37	41394	42076	42758	43439	44121	44802	45483	46164	46845	47526
38	48207	48887	49568	50248	50929	51609	52289	52969	53649	54329
39	55009	55688	56368	57047	57726	58405	59085	59764	60442	61121
40	61800	62478	63157	63835	64513	65191	65869	66547	67225	67903
41	68580	69258	69935	70612	71290	71967	72644	73320	73997	74674
42	75350	76027	76703	77379	78055	78731	79407	80083	80759	81434
43	82110	82785	83460	84136	84811	85486	86160	86835	87510	88184
44	88859	89533	90207	90881	91555	92229	92903	93577	94250	94924
45	95597	96270	96944	97617	98290	98962	99635	8100308	8100980	8101653
46	8102325	8102997	8103670	8104342	8105013	8105685	8106357	07029	07700	08372
47	09043	09714	10385	11056	11727	12398	13068	13739	14409	15080
48	15750	16420	17090	17760	18430	19100	19769	20439	21108	21778
49	22447	23116	23785	24454	25123	25792	26460	27129	27797	28465

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
650	8129134	8129802	8130470	8131138	8131805	8132473	8133141	8133808	8134475	8135143
51	35810	36477	37144	37811	38478	39144	39811	40477	41144	41810
52	42476	43142	43808	44474	45140	45805	46471	47136	47801	48467
53	49132	49797	50462	51127	51791	52456	53120	53785	54449	55113
54	55777	56441	57105	57769	58433	59097	59760	60423	61087	61750
55	62413	63076	63739	64402	65064	65727	66389	67052	67714	68376
56	69038	69700	70362	71024	71686	72347	73009	73670	74331	74993
57	75654	76315	76976	77636	78297	78958	79618	80278	80939	81599
58	82259	82919	83579	84239	84898	85558	86217	86877	87536	88195
59	88854	89513	90172	90831	91489	92148	92806	93465	94123	94781
60	95439	96097	96755	97413	98071	98728	99386	8200043	8200700	8201358
61	8202015	8202672	8203328	8203985	8204642	8205298	8205955	06611	07268	07924
62	08580	09236	09892	10548	11203	11859	12514	13170	13825	14480
63	15135	15790	16445	17100	17755	18409	19064	19718	20372	21027
64	21681	22335	22989	23643	24296	24950	25603	26257	26910	27563
65	28216	28869	29522	30175	30828	31481	32133	32786	33438	34090
66	34742	35394	36046	36698	37350	38002	38653	39305	39956	40607
67	41258	41909	42560	43211	43862	44513	45163	45814	46464	47114
68	47765	48415	49065	49715	50364	51014	51664	52313	52963	53612
69	54261	54910	55559	56208	56857	57506	58154	58803	59451	60100
70	60748	61396	62044	62692	63340	63988	64635	65283	65931	66578
71	67225	67872	68519	69166	69813	70460	71107	71753	72400	73046
72	73693	74339	74985	75631	76277	76923	77569	78214	78860	79505
73	80151	80796	81441	82086	82731	83376	84021	84665	85310	85955
74	86599	87243	87887	88532	89176	89820	90463	91107	91751	92394
75	93038	93681	94324	94967	95611	96254	96896	97539	98182	98824
76	99467	8300109	8300752	8301394	8302036	8302678	8303320	8303962	8304604	8305245
77	8305887	06528	07169	07811	08452	09093	09734	10375	11016	11656
78	12297	12937	13578	14218	14858	15499	16139	16778	17418	18058
79	18698	19337	19977	20616	21255	21895	22534	23173	23812	24450
80	25089	25728	26366	27005	27643	28281	28919	29558	30195	30833
81	31471	32109	32746	33384	34021	34659	35296	35933	36570	37207
82	37844	38480	39117	39754	40390	41027	41663	42299	42935	43571
83	44207	44843	45479	46114	46750	47385	48021	48656	49291	49926
84	50561	51196	51831	52465	53100	53735	54369	55003	55638	56272
85	56906	57540	58174	58807	59441	60075	60708	61341	61975	62608
86	63241	63874	64507	65140	65773	66405	67038	67670	68303	68935
87	69567	70199	70832	71463	72095	72727	73359	73990	74622	75253
88	75884	76516	77147	77778	78409	79039	79670	80301	80931	81562
89	82192	82822	83453	84083	84713	85343	85973	86602	87232	87861
90	88491	89120	89750	90379	91008	91637	92266	92895	93523	94152
91	94780	95409	96037	96666	97294	97922	98550	99178	99806	8400433
92	8401061	8401688	8402316	8402943	8403571	8404198	8404825	8405452	8406079	06706
93	07332	07959	08586	09212	09838	10465	11091	11717	12343	12969
94	13595	14220	14846	15472	16097	16723	17348	17973	18598	19223
95	19848	20473	21098	21722	22347	22971	23596	24220	24844	25468
96	26092	26716	27340	27964	28588	29211	29835	30458	31081	31705
97	32328	32951	33574	34197	34819	35442	36065	36687	37310	37932
98	38554	39176	39798	40420	41042	41664	42286	42907	43529	44150
99	44772	45393	46014	46635	47256	47877	48498	49119	49739	50360

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700	8450980	8451601	8452221	8452841	8453461	8454081	8454701	8455321	8455941	8456561
1	57180	57800	58419	59038	59658	60277	60896	61515	62134	62752
2	63371	63990	64608	65227	65845	66463	67081	67700	68318	68935
3	69553	70171	70789	71406	72024	72641	73258	73876	74493	75110
4	75727	76343	76960	77577	78193	78810	79426	80043	80659	81275
5	81891	82507	83123	83739	84355	84970	85586	86201	86817	87432
6	88047	88662	89277	89892	90507	91122	91736	92351	92965	93580
7	94194	94808	95423	96037	96651	97264	97878	98492	99106	99719
8	8500333	8500946	8501559	8502172	8502786	8503399	8504011	8504624	8505237	8505850
9	06462	07075	07687	08300	08912	09524	10136	10748	11360	11972
10	12583	13195	13807	14418	15030	15641	16252	16863	17474	18085
11	18696	19307	19917	20528	21139	21749	22359	22970	23580	24190
12	24800	25410	26020	26629	27239	27849	28458	29068	29677	30286
13	30895	31504	32113	32722	33331	33940	34548	35157	35765	36374
14	36982	37590	38198	38807	39414	40022	40630	41238	41845	42453
15	43060	43668	44275	44882	45489	46096	46703	47310	47917	48524
16	49130	49737	50343	50950	51556	52162	52768	53374	53980	54586
17	55192	55797	56403	57008	57614	58219	58824	59429	60035	60640
18	61244	61849	62454	63059	63663	64268	64872	65476	66081	66685
19	67289	67893	68497	69101	69704	70308	70912	71515	72118	72722
20	73325	73928	74531	75134	75737	76340	76943	77545	78148	78750
21	79353	79955	80557	81159	81761	82363	82965	83567	84169	84770
22	85372	85973	86575	87176	87777	88379	88980	89581	90181	90782
23	91383	91984	92584	93185	93785	94385	94986	95586	96186	96786
24	97386	97985	98585	99185	99784	8600384	8600983	8601583	8602182	8602781
25	8603380	8603979	8604578	8605177	8605776	06374	06973	07571	08170	08768
26	09366	09964	10562	11160	11758	12356	12954	13552	14149	14747
27	15344	15941	16539	17136	17733	18330	18927	19524	20121	20717
28	21314	21910	22507	23103	23699	24296	24892	25488	26084	26680
29	27275	27871	28467	29062	29658	30253	30848	31443	32039	32634
30	33229	33823	34418	35013	35608	36202	36797	37391	37985	38580
31	39174	39768	40362	40956	41550	42143	42737	43331	43924	44517
32	45111	45704	46297	46890	47483	48076	48669	49262	49855	50447
33	51040	51632	52225	52817	53409	54001	54593	55185	55777	56369
34	56961	57552	58144	58735	59327	59918	60509	61100	61691	62282
35	62873	63464	64055	64646	65236	65827	66417	67008	67598	68188
36	68778	69368	69958	70548	71138	71728	72317	72907	73496	74086
37	74675	75264	75853	76442	77031	77620	78209	78798	79387	79975
38	80564	81152	81740	82329	82917	83505	84093	84681	85269	85857
39	86444	87032	87620	88207	88794	89382	89969	90556	91143	91730
40	92317	92904	93491	94077	94664	95251	95837	96423	97010	97596
41	98182	98768	99354	99940	8700526	8701112	8701697	8702283	8702868	8703454
42	8704039	8704624	8705210	8705795	06380	06965	07549	08134	08719	09304
43	09888	10473	11057	11641	12226	12810	13394	13978	14562	15146
44	15729	16313	16897	17480	18064	18647	19230	19814	20397	20980
45	21563	22146	22728	23311	23894	24477	25059	25641	26224	26806
46	27388	27970	28552	29134	29716	30298	30880	31462	32043	32625
47	33206	33787	34369	34950	35531	36112	36693	37274	37855	38435
48	39016	39597	40177	40757	41338	41918	42498	43078	43658	44238
49	44818	45398	45978	46557	47137	47716	48296	48875	49454	50034

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750	8750613	8751192	8751771	8752349	8752928	8753507	8754086	8754664	8755243	8755821
51	56399	56978	57556	58134	58712	59290	59868	60446	61023	61601
52	62178	62756	63333	63911	64488	65065	65642	66219	66796	67373
53	67950	68526	69103	69680	70256	70833	71409	71985	72561	73137
54	73713	74289	74865	75441	76017	76592	77168	77743	78319	78894
55	79470	80045	80620	81195	81770	82345	82919	83494	84069	84643
56	85218	85792	86367	86941	87515	88089	88663	89237	89811	90385
57	90959	91532	92106	92680	93253	93826	94400	94973	95546	96119
58	96692	97265	97838	98411	98983	99556	8800128	8800701	8801272	8801846
59	8802418	8802990	8803562	8804134	8804706	8805278	05850	06421	06993	07564
60	08136	08707	09279	09850	10421	10992	11563	12134	12705	13276
61	13847	14417	14988	15558	16129	16699	17269	17840	18410	18980
62	19550	20120	20689	21259	21829	22398	22968	23537	24107	24676
63	25245	25815	26384	26953	27522	28090	28659	29228	29797	30365
64	30934	31502	32070	32639	33207	33775	34343	34911	35479	36047
65	36614	37182	37750	38317	38885	39452	40019	40586	41154	41721
66	42288	42855	43421	43988	44555	45122	45688	46255	46821	47387
67	47954	48520	49086	49652	50218	50784	51350	51915	52481	53047
68	53612	54178	54743	55308	55874	56439	57004	57569	58134	58699
69	59263	59828	60393	60957	61522	62086	62651	63215	63779	64343
70	64907	65471	66035	66599	67163	67726	68290	68854	69417	69980
71	70544	71107	71670	72233	72796	73359	73922	74485	75048	75610
72	76173	76736	77298	77860	78423	78985	79547	80109	80671	81233
73	81795	82357	82918	83480	84042	84603	85165	85726	86287	86848
74	87410	87971	88532	89093	89653	90214	90775	91336	91896	92457
75	93017	93577	94138	94698	95258	95818	96378	96938	97498	98058
76	98617	99177	99736	8900296	8900855	8901415	8901974	8902533	8903092	8903651
77	8904210	8904769	8905328	05887	06445	07004	07563	08121	08679	09238
78	09796	10354	10912	11470	12028	12586	13144	13702	14259	14817
79	15375	15932	16489	17047	17604	18161	18718	19275	19832	20389
80	20946	21503	22059	22616	23173	23729	24285	24842	25398	25954
81	26510	27066	27622	28178	28734	29290	29846	30401	30957	31512
82	32068	32623	33178	33733	34288	34843	35398	35953	36508	37063
83	37618	38172	38727	39281	39836	40390	40944	41498	42053	42607
84	43161	43715	44268	44822	45376	45929	46483	47037	47590	48143
85	48697	49250	49803	50356	50909	51462	52015	52568	53120	53673
86	54225	54778	55330	55883	56435	56987	57539	58092	58644	59195
87	59747	60299	60851	61403	61954	62506	63057	63608	64160	64711
88	65262	65813	66364	66915	67466	68017	68568	69118	69669	70220
89	70770	71320	71871	72421	72971	73521	74071	74621	75171	75721
90	76271	76821	77370	77920	78469	79019	79568	80117	80667	81216
91	81765	82314	82863	83412	83960	84509	85058	85606	86155	86703
92	87252	87800	88348	88897	89445	89993	90541	91089	91636	92184
93	92732	93279	93827	94375	94922	95469	96017	96564	97111	97658
94	98205	98752	99299	99846	9000392	9000939	9001486	9002032	9002579	9003125
95	9003671	9004218	9004764	9005310	05856	06402	06948	07494	08039	08585
96	09131	09676	10222	10767	11313	11858	12403	12948	13493	14038
97	14583	15128	15673	16218	16762	17307	17851	18396	18940	19485
98	20029	20573	21117	21661	22205	22749	23293	23837	24381	24924
99	25468	26011	26555	27098	27641	28185	28728	29271	29814	30357

SEVEN-FIGURE LOGARITHMS

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800	9030900	9031443	9031985	9032528	9033071	9033613	9034156	9034698	9035241	9035783
1	36325	36867	37409	37951	38493	39035	39577	40119	40661	41202
2	41744	42285	42827	43368	43909	44450	44992	45533	46074	46615
3	47155	47696	48237	48778	49318	49859	50399	50940	51480	52020
4	52560	53101	53641	54181	54721	55260	55800	56340	56880	57419
5	57959	58498	59038	59577	60116	60655	61195	61734	62273	62812
6	63350	63889	64428	64967	65505	66044	66582	67121	67659	68197
7	68735	69273	69812	70350	70887	71425	71963	72501	73038	73576
8	74114	74651	75188	75726	76263	76800	77337	77874	78411	78948
9	79485	80022	80559	81095	81632	82169	82705	83241	83778	84314
10	84850	85386	85922	86458	86994	87530	88066	88602	89137	89673
11	90209	90744	91279	91815	92350	92885	93420	93955	94490	95025
12	95560	96095	96630	97165	97699	98234	98768	99303	99837	9100371
13	9100905	9101440	9101974	9102508	9103042	9103576	9104109	9104643	9105177	05710
14	06244	06778	07311	07844	08378	08911	09444	09977	10510	11043
15	11576	12109	12642	13174	13707	14240	14772	15305	15837	16369
16	16902	17434	17966	18498	19030	19562	20094	20626	21157	21689
17	22221	22752	23284	23815	24346	24878	25409	25940	26471	27002
18	27533	28064	28595	29126	29656	30187	30717	31248	31778	32309
19	32839	33369	33899	34430	34960	35490	36019	36549	37079	37609
20	38139	38668	39198	39727	40257	40786	41315	41844	42373	42903
21	43432	43961	44489	45018	45547	46076	46604	47133	47661	48190
22	48718	49246	49775	50303	50831	51359	51887	52415	52943	53471
23	53998	54526	55054	55581	56109	56636	57163	57691	58218	58745
24	59272	59799	60326	60853	61380	61907	62433	62960	63487	64013
25	64539	65066	65592	66118	66645	67171	67697	68223	68749	69275
26	69800	70326	70852	71378	71903	72429	72954	73479	74005	74530
27	75055	75580	76105	76630	77155	77680	78205	78730	79254	79779
28	80303	80828	81352	81877	82401	82925	83449	83973	84497	85021
29	85545	86069	86593	87117	87640	88164	88687	89211	89734	90258
30	90781	91304	91827	92350	92873	93396	93919	94442	94965	95488
31	96010	96533	97055	97578	98100	98623	99145	99667	9200189	9200711
32	9201233	9201755	9202277	9202799	9203321	9203842	9204364	9204886	05407	05929
33	06450	06971	07493	08014	08535	09056	09577	10098	10619	11140
34	11661	12181	12702	13222	13743	14263	14784	15304	15824	16345
35	16865	17385	17905	18425	18945	19465	19984	20504	21024	21543
36	22063	22582	23102	23621	24140	24659	25179	25698	26217	26736
37	27255	27773	28292	28811	29330	29848	30367	30885	31404	31922
38	32440	32958	33477	33995	34513	35031	35549	36066	36584	37102
39	37620	38137	38655	39172	39690	40207	40724	41242	41759	42276
40	42793	43310	43827	44344	44860	45377	45894	46410	46927	47444
41	47960	48476	48993	49509	50025	50541	51057	51573	52089	52605
42	53121	53637	54152	54668	55184	55699	56215	56730	57245	57761
43	58276	58791	59306	59821	60336	60851	61366	61880	62395	62910
44	63424	63939	64453	64968	65482	65997	66511	67025	67539	68053
45	68567	69081	69595	70109	70622	71136	71650	72163	72677	73190
46	73704	74217	74730	75243	75757	76270	76783	77296	77808	78321
47	78834	79347	79859	80372	80885	81397	81909	82422	82934	83446
48	83959	84471	84983	85495	86007	86518	87030	87542	88054	88565
49	89077	89588	90100	90611	91123	91634	92145	92656	93167	93678

SEVEN-FIGURE LOGARITHMS

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850	9294189	9294700	9295211	9295722	9296233	9296743	9297254	9297764	9298275	9298785
51	99296	99806	9300316	9300826	9301336	9301847	9302357	9302866	9303376	9303886
52	9304396	9304906	05415	05925	06434	06944	07453	07963	08472	08981
53	09490	09999	10508	11017	11526	12035	12544	13053	13562	14070
54	14579	15087	15596	16104	16612	17121	17629	18137	18645	19153
55	19661	20169	20677	21185	21692	22200	22708	23215	23723	24230
56	24738	25245	25752	26259	26767	27274	27781	28288	28795	29301
57	29808	30315	30822	31328	31835	32341	32848	33354	33860	34367
58	34873	35379	35885	36391	36897	37403	37909	38415	38920	39426
59	39932	40437	40943	41448	41953	42459	42964	43469	43974	44479
60	44085	44589	45094	45599	46104	46609	47113	47618	48123	48627
61	50032	50536	51040	51544	52049	52553	53057	53561	54065	54569
62	55073	55576	56080	56584	57087	57591	58095	58598	59101	59605
63	60108	60611	61114	61617	62120	62623	63126	63629	64132	64635
64	65137	65640	66143	66645	67148	67650	68152	68655	69157	69659
65	70161	70663	71165	71667	72169	72671	73172	73674	74176	74677
66	75179	75680	76182	76683	77184	77686	78187	78688	79189	79690
67	80191	80692	81193	81693	82194	82695	83195	83696	84196	84697
68	85197	85698	86198	86698	87198	87698	88198	88698	89198	89698
69	90198	90697	91197	91697	92196	92696	93195	93695	94194	94693
70	95193	95692	96191	96690	97189	97688	98187	98685	99184	99683
71	9400182	9400680	9401179	9401677	9402176	9402674	9403172	9403670	9404169	9404667
72	05165	05663	06161	06659	07157	07654	08152	08650	09147	09645
73	10142	10640	11137	11635	12132	12629	13126	13623	14120	14617
74	15114	15611	16108	16605	17101	17598	18095	18591	19088	19584
75	20081	20577	21073	21569	22065	22562	23058	23553	24049	24545
76	25041	25537	26032	26528	27024	27519	28015	28510	29005	29501
77	29996	30491	30986	31481	31976	32471	32966	33461	33956	34450
78	34945	35440	35934	36429	36923	37418	37912	38406	38900	39395
79	39889	40383	40877	41371	41865	42358	42852	43346	43840	44333
80	44827	45320	45814	46307	46800	47294	47787	48280	48773	49266
81	49759	50252	50745	51238	51730	52223	52716	53208	53701	54193
82	54686	55178	55671	56163	56655	57147	57639	58131	58623	59115
83	59607	60099	60591	61082	61574	62066	62557	63049	63540	64031
84	64523	65014	65505	65996	66487	66978	67469	67960	68451	68942
85	69433	69923	70414	70905	71395	71886	72376	72866	73357	73847
86	74337	74827	75317	75807	76297	76787	77277	77767	78257	78747
87	79236	79726	80215	80705	81194	81684	82173	82662	83151	83641
88	84130	84619	85108	85597	86085	86574	87063	87552	88040	88529
89	89018	89506	89995	90483	90971	91460	91948	92436	92924	93412
90	93900	94388	94876	95364	95852	96339	96827	97315	97802	98290
91	98777	99264	99752	9500239	9500726	9501213	9501701	9502188	9502675	9503162
92	9503649	9504135	9504622	05109	05596	06082	06569	07055	07542	08028
93	08515	09001	09487	09973	10459	10946	11432	11918	12404	12889
94	13375	13861	14347	14832	15318	15803	16289	16774	17260	17745
95	18230	18716	19201	19686	20171	20656	21141	21626	22111	22595
96	23080	23565	24049	24534	25018	25503	25987	26472	26956	27440
97	27924	28409	28893	29377	29861	30345	30828	31312	31796	32280
98	32763	33247	33731	34214	34697	35181	35664	36147	36631	37114
99	37597	38080	38563	39046	39529	40012	40494	40977	41460	41943

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
900	9542425	9542908	9543390	9543873	9544355	9544837	9545319	9545802	9546284	9546766
1	47248	47730	48212	48694	49176	49657	50139	50621	51102	51584
2	52065	52547	53028	53510	53991	54472	54953	55434	55916	56397
3	56878	57358	57839	58320	58801	59282	59762	60243	60723	61204
4	61684	62165	62645	63125	63606	64086	64566	65046	65526	66006
5	66486	66966	67445	67925	68405	68885	69364	69844	70323	70803
6	71282	71761	72241	72720	73199	73678	74157	74636	75115	75594
7	76073	76552	77030	77509	77988	78466	78945	79423	79902	80380
8	80858	81337	81815	82293	82771	83249	83727	84205	84683	85161
9	85639	86117	86594	87072	87549	88027	88505	88982	89459	89937
10	90414	90891	91368	91845	92322	92800	93276	93753	94230	94707
11	95184	95660	96137	96614	97090	97567	98043	98520	98996	99472
12	99948	9600425	9600901	9601377	9601853	9602329	9602805	9603281	9603756	9604232
13	9604708	05183	05659	06135	06610	07086	07561	08036	08512	08987
14	09462	09937	10412	10887	11362	11837	12312	12787	13262	13736
15	14211	14686	15160	15635	16109	16583	17058	17532	18006	18481
16	18955	19429	19903	20377	20851	21325	21799	22272	22746	23220
17	23693	24167	24640	25114	25587	26061	26534	27007	27481	27954
18	28427	28900	29373	29846	30319	30792	31264	31737	32210	32683
19	33155	33628	34100	34573	35045	35517	35990	36462	36934	37406
20	37878	38350	38822	39294	39766	40238	40710	41181	41653	42125
21	42596	43068	43539	44011	44482	44953	45425	45896	46367	46838
22	47309	47780	48251	48722	49193	49664	50135	50605	51076	51546
23	52017	52488	52958	53428	53899	54369	54839	55309	55780	56250
24	56720	57190	57660	58130	58599	59069	59539	60009	60478	60948
25	61417	61887	62356	62826	63295	63764	64233	64703	65172	65641
26	66110	66579	67048	67517	67985	68454	68923	69392	69860	70329
27	70797	71266	71734	72203	72671	73139	73607	74076	74544	75012
28	75480	75948	76416	76884	77351	77819	78287	78754	79222	79690
29	80157	80625	81092	81559	82027	82494	82961	83428	83895	84362
30	84829	85296	85763	86230	86697	87164	87630	88097	88564	89030
31	89497	89963	90430	90896	91362	91829	92295	92761	93227	93693
32	94159	94625	95091	95557	96023	96488	96954	97420	97885	98351
33	98816	99282	99747	9700213	9700678	9701143	9701608	9702074	9702539	9703004
34	9703469	9703934	9704399	04863	05328	05793	06258	06722	07187	07652
35	08116	08581	09045	09509	09974	10438	10902	11366	11830	12294
36	12758	13222	13686	14150	14614	15078	15542	16005	16469	16932
37	17396	17859	18323	18786	19249	19713	20176	20639	21102	21565
38	22028	22491	22954	23417	23880	24343	24805	25268	25731	26193
39	26656	27118	27581	28043	28506	28968	29430	29892	30354	30816
40	31279	31741	32202	32664	33126	33588	34050	34511	34973	35435
41	35896	36358	36819	37281	37742	38203	38664	39126	39587	40048
42	40509	40970	41431	41892	42353	42814	43274	43735	44196	44656
43	45117	45577	46038	46498	46959	47419	47879	48340	48800	49260
44	49720	50180	50640	51100	51560	52020	52479	52939	53399	53858
45	54318	54778	55237	55697	56156	56615	57075	57534	57993	58452
46	58911	59370	59829	60288	60747	61206	61665	62124	62582	63041
47	63500	63958	64417	64875	65334	65792	66251	66709	67167	67625
48	68083	68541	69000	69458	69915	70373	70831	71289	71747	72204
49	72662	73120	73577	74035	74492	74950	75407	75864	76322	76779

SEVEN-FIGURE LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
950	9777236	9777693	9778150	9778607	9779064	9779521	9779978	9780435	9780892	9781348
51	81805	82262	82718	83175	83631	84088	84544	85001	85457	85913
52	86369	86826	87282	87738	88194	88650	89106	89562	90017	90473
53	90929	91385	91840	92296	92751	93207	93662	94118	94573	95028
54	95484	95939	96394	96849	97304	97759	98214	98669	99124	99579
55	9800034	9800488	9800943	9801398	9801852	9802307	9802761	9803216	9803670	9804125
56	04579	05033	05487	05942	06396	06850	07304	07758	08212	08666
57	09119	09573	10027	10481	10934	11388	11841	12295	12748	13202
58	13655	14108	14562	15015	15468	15921	16374	16827	17280	17733
59	18186	18639	19092	19544	19997	20450	20902	21355	21807	22260
60	22712	23165	23617	24069	24522	24974	25426	25878	26330	26782
61	27234	27686	28138	28589	29041	29493	29945	30396	30848	31299
62	31751	32202	32654	33105	33556	34007	34459	34910	35361	35812
63	36263	36714	37165	37616	38066	38517	38968	39419	39869	40320
64	40770	41221	41671	42122	42572	43022	43473	43923	44373	44823
65	45273	45723	46173	46623	47073	47523	47973	48422	48872	49322
66	49771	50221	50670	51120	51569	52019	52468	52917	53366	53816
67	54265	54714	55163	55612	56061	56510	56959	57407	57856	58305
68	58754	59202	59651	60099	60548	60996	61445	61893	62341	62790
69	63238	63686	64134	64582	65030	65478	65926	66374	66822	67270
70	67717	68165	68613	69060	69508	69955	70403	70850	71298	71745
71	72192	72640	73087	73534	73981	74428	74875	75322	75769	76216
72	76663	77109	77556	78003	78450	78896	79343	79789	80236	80682
73	81128	81575	82021	82467	82913	83360	83806	84252	84698	85144
74	85590	86035	86481	86927	87373	87818	88264	88710	89155	89601
75	90046	90492	90937	91382	91828	92273	92718	93163	93608	94053
76	94498	94943	95388	95833	96278	96722	97167	97612	98057	98501
77	98946	99390	99835	9900279	9900723	9901168	9901612	9902056	9902500	9902944
78	9903389	9903833	9904277	04721	05164	05608	06052	06496	06940	07383
79	07827	08271	08714	09158	09601	10044	10488	10931	11374	11818
80	12261	12704	13147	13590	14033	14476	14919	15362	15805	16247
81	16690	17133	17575	18018	18461	18903	19345	19788	20230	20673
82	21115	21557	21999	22441	22884	23326	23768	24210	24651	25093
83	25535	25977	26419	26860	27302	27744	28185	28627	29068	29510
84	29951	30392	30834	31275	31716	32157	32598	33039	33480	33921
85	34362	34803	35244	35685	36126	36566	37007	37448	37888	38329
86	38769	39210	39650	40090	40531	40971	41411	41851	42291	42731
87	43172	43612	44051	44491	44931	45371	45811	46251	46690	47130
88	47569	48009	48448	48888	49327	49767	50206	50645	51085	51524
89	51963	52402	52841	53280	53719	54158	54597	55036	55474	55913
90	56352	56791	57229	57668	58106	58545	58983	59422	59860	60298
91	60737	61175	61613	62051	62489	62927	63365	63803	64241	64679
92	65117	65554	65992	66430	66868	67305	67743	68180	68618	69055
93	69492	69930	70367	70804	71242	71679	72116	72553	72990	73427
94	73864	74301	74738	75174	75611	76048	76485	76921	77358	77794
95	78231	78667	79104	79540	79976	80413	80849	81285	81721	82157
96	82593	83029	83465	83901	84337	84773	85209	85645	86080	86516
97	86952	87387	87823	88258	88694	89129	89564	90000	90435	90870
98	91305	91741	92176	92611	93046	93481	93916	94350	94785	95220
99	95665	96099	96524	96959	97393	97828	98262	98697	99131	99566

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ANSWERS

EXERCISES 1A. (Page 5)

1. $17\frac{1}{8}$ d. 2. $178\frac{3}{4}$ francs. 3. (i) 13s. 10d.; (ii) £4 6s. 4d.
4. (i) 3 cwt. 26 lb.; (ii) 3 tons 13 cwt. 1 qr. 1 lb. 5. (i) $\frac{1}{60}$; (ii) $\frac{1}{23}$.
6. $\frac{1}{48}$; £3 2s. 4d. 7. £26 18s. 6d. 8. $\frac{1}{19}$; £2 13s. 1d.
9. (i) $\frac{2}{7}$; (ii) $\frac{1}{21}$; (iii) $\frac{1}{17}$. 10. $\frac{2}{9}$. 11. 247 dollars.
12. 1375 francs. 13. $\frac{3}{28}$. 14. B, £3344, C, £1596; $\frac{4}{19}$. 15. $\frac{1}{200}$.

EXERCISES 1B. (Page 14)

1. $9\frac{1}{8}$. 2. 6s. 11d. 3. $\frac{8}{9}$; 245 kgm. 4. $\frac{7}{8}$.
5. $32\frac{1}{4}$ mi. per hr. 6. £7 10s. 7. 5 miles.
8. $\frac{2}{3}$ ton. 9. (i) £ $\frac{9}{80}$; (ii) $\frac{8}{140}$ ton; £4 1s. 8d. per ton.
10. $\frac{1}{21}$; 4 tons. 11. $\frac{2}{3}$ ton; £4 15s. 10d.
12. $\frac{5}{12}$ ton; $53\frac{1}{3}$ metric tons. 13. £25 13s. 4d. per ton.
14. £5 13s. 4d. per ton. 15. £1 10s. 6d. 16. £32 13s. 4d. per ton.
17. $\frac{3}{7}$. 18. $\frac{1}{21}$. 19. (i) $1\frac{1}{4}$; (ii) $\frac{1}{3}$. 20. 6s. 4d. 21. $\frac{3}{4}$.
22. $\frac{8}{17}$. 23. £3 15s. per week. 24. £6 0s. 9d.
25. Average profit = £5236; (i) $\frac{1}{9}$; (ii) $1\frac{1}{8}$.
26. A pays £24 3s. 1d., B pays £24 3s. 9d.; B pays 8d. more than A.
27. £15 19s. 6d. 28. 1s. $5\frac{1}{2}$ d. per lb. 29. 1s. 5d. per lb.
30. Interest = 8s. 2d.; $\frac{1}{90}$.

EXERCISES 2A. (Page 28)

1. (i) £0·8292; (ii) 0·8393 cwt. 2. (i) £0·472; (ii) £0·579.
3. (i) £6 15s. $1\frac{1}{2}$ d.; (ii) £96 6s. 6d.; (iii) £8 3s. 3d.
4. (i) £0·119; (ii) £0·390; (iii) £0·608.
5. (i) 12 cwt. 1 qr. 11 lb.; (ii) 10 cwt. 22 lb. 6. 2·14002.
7. (i) 14s. 9d.; (ii) 1s. 4d.
8. (i) £0·9125, (ii) 0·6844 ton; £5 2s. 8d. 9. 63·5.
10. 4030·3. 11. 3058·87. 12. 48·153.
13. 3·86. 14. 102·74. 15. 35.
16. £4 12s. 8d. 17. 97 francs. 18. 79·7 metric tons.
19. 0·621 mile. 20. £485 17s. 1d. 21. 3·53 cu. ft.
22. 131·1 francs per kgm. 23. (i) $\frac{3}{91}$; (ii) 0·407.
24. 0·7857 cwt., £27 10s. 2d. 25. £377 1s. 5d.
26. 2·47 acres.

EXERCISES 2B. (Page 34)

1. 0.2594.
2. £26 3s. 1d.
3. £6 18s.
4. 11s. $10\frac{1}{2}$ d.
5. £16 4s. 11d.
6. 192.9 tons.
7. (i) 18.73 per 100; (ii) 1.5 and 1.8 per acre.
8. 0.673.
9. 7.709.
10. 10.83.
11. 6.75.
12. 13s. $5\frac{1}{2}$ d.
13. 0.162.
14. £0.35 = 7s.
15. 8s. 4d.
16. £159,865.
17. 53.75 dollars.
18. (i) 22.76 men; (ii) 18.61 women; (iii) 12.08.
19. £26 17s. 8d. per ton.
20. £10.1.
21. 11s. $9\frac{1}{2}$ d. in the £.
22. 1s. $2\frac{1}{3}$ d. per lb.
23. 15.7 cwt. per acre.

EXERCISES 3. Section I. (Page 49)

1. £1843 16s. 4d.
2. 1760 yd. = 1 mile.
3. 100 tons 18 cwt. 3 qr. 19 lb.
4. (i)

	£	s.	d.
(a)	394	12	10
(b)	130	3	3
(c)	297	9	4
(d)	427	12	6
(e)	301	1	3
(f)	64	14	3
(g)	359	16	1
5. (i)

	£	s.	d.
	476	17	5
	453	11	11
	332	7	1
	299	8	6
	380	11	11
	366	19	4
	518	9	5
6. (i)

	£	s.	d.
	503	19	9
	222	7	3
	348	17	11
	212	13	11
	407	7	1
	666	3	1
	207	12	6
- (ii)

A.	691	19	4
B.	594	12	9
C.	688	17	5
- (ii)

	673	16	0
	1196	9	9
	957	19	10
- (ii)

	898	3	7
	860	16	5
	810	1	6
- (iii)

G.T.	1975	9	6
------	------	---	---
- (iii)

	2828	5	7
--	------	---	---
- (iii)

	2569	1	6
--	------	---	---
7. (i)

	£	s.	d.
(a)	2494	18	7
(b)	2451	8	3
(c)	2235	1	5
(d)	2468	19	1
(e)	3124	1	8
(f)	2236	3	8
(g)	2671	5	8
(h)	2651	8	4
- (ii)

A.	8542	0	4
B.	6063	19	8
C.	5727	6	8
- (iii)

G.T.	20333	6	8
------	-------	---	---
8. (i)

	£	s.	d.
	4635	1	$0\frac{1}{2}$
	5060	4	$1\frac{3}{4}$
	7981	7	1
	1970	11	$8\frac{1}{4}$
	3679	17	$5\frac{1}{2}$
	2979	16	$7\frac{3}{4}$
	2063	17	3
	6320	17	$9\frac{1}{2}$
- (ii)

	15294	19	1
	17085	19	4
	2310	14	$8\frac{1}{2}$
- (iii)

	34691	13	$1\frac{1}{4}$
--	-------	----	----------------

9.	(i)		
	£	s.	d.
(a)	7065	17	5
(b)	293	19	6
(c)	1363	12	2
(d)	7443	10	10
(e)	3073	2	6
(f)	6192	18	0
(g)	863	13	11
(h)	2281	0	3
(i)	2014	7	2
(k)	4265	6	2
(l)	4888	12	2
(m)	3321	13	2

	(ii)		
A.	16461	2	6
B.	13667	15	5
C.	12938	15	4

	(iii)		
G.T.	43067	13	3

10.	(i)		
	£	s.	d.
	14831	15	0
	8657	12	1
	7394	15	1
	3901	19	0
	2051	18	0
	15693	4	5
	1480	4	1
	9268	11	7
	13077	7	5
	17984	13	11
	2710	11	4
	6578	12	10

	(ii)		
	49979	4	6
	45792	13	1
	7859	7	2

	(iii)		
	103631	4	9

Section II. (Page 52)

11.	£	s.	d.
	-	9	9
	-	13	6
	-	5	3
	<u>1</u>	<u>8</u>	<u>6</u>

12.	£	s.	d.
	-	6	$6\frac{3}{4}$
	-	2	$5\frac{3}{4}$
	-	-	$6\frac{1}{2}$
	-	1	$4\frac{1}{4}$
	<u>-</u>	<u>10</u>	<u>$11\frac{1}{4}$</u>

13.	£	s.	d.
	12	16	8
	1	13	4
	-	17	0
	-	12	10
	-	13	6
	16	13	4
	-	8	4
	<u>16</u>	<u>5</u>	<u>0</u>

Less

14.	£	s.	d.
	67	17	$10\frac{1}{2}$
	20	15	$7\frac{1}{2}$
	19	17	6
	<u>108</u>	<u>11</u>	<u>0</u>

15. (a) £2 13s. 8d. (b) $4\frac{1}{2}$ d. (c) 1 cwt. 1 qr. 16 lb. (d) £10 2s. 2d.
 16. 13s. 2d. 17. (i) £330 12s. 7d., (ii) £5 18s. 7d.
 18. £90 5s. 2d. 19. 411 yd. 20. 52 tons 12 cwt. 3 qr. 25 lb.
 21. 3 cwt. 2 qr. 16 lb. = 408 lb.; £6 7s. 6d. 22. £1 10s. 10d.
 23. £111. 24. £1 = 176.64 francs. 25. 40 lb.
 26. 4 cwt. 3 qr. 17 lb. 27. 4 dollars $87\frac{1}{2}$ cents. 28. 24.25 pence.
 29. £3 11s. 30. (i) £1 = 4.867 dollars; (ii) 49.3 pence.
 31. £1 = 23.36 belgas.

EXERCISES 4A. (Page 59)

- | | | | |
|-------------|-------------|-------------|-------------|
| 1. £0.933. | 2. £3.342. | 3. £3.871. | 4. £5.579. |
| 5. £0.885. | 6. £8.931. | 7. £0.390. | 8. £1.770. |
| 9. £15.793. | 10. £1.885. | 11. £0.941. | 12. £0.893. |

- | | | | |
|------------------|------------------|-----------------|------------------|
| 13. £3-773. | 14. £21-674. | 15. £0-619. | 16. £0-316. |
| 17. £13-739. | 18. £0-923. | 19. £6-093. | 20. £1-061. |
| 21. £5 5s. 9d. | 22. 10s. 11d. | 23. £3 10s. 6d. | 24. 1s. 8d. |
| 25. £6 15s. 11d. | 26. 13s. 7d. | 27. 10s. 11d. | 28. £3 2s. 5d. |
| 29. £15 6s. 11d. | 30. £4 14s. 7d. | 31. 1s. 4½d. | 32. 17s. 9¾d. |
| 33. £5 17s. 6¼d. | 34. £13 12s. 2d. | 35. £1 6s. 5¾d. | 36. £1 11s. 6¾d. |
| 37. 14s. 11½d. | 38. £2 17s. 5¾d. | 39. £1 1s. 7¼d. | 40. £7 19s. 7½d. |

EXERCISES 4B. (Page 67)

- | | | |
|--|-----------------------------------|-----------------|
| 1. £0-5958333 ; £0-5958. | 2. £0-8708333 ; £0-8708. | |
| 3. £1-1791667 ; £1-1792. | 4. £8-9541667 ; £8-9542. | |
| 5. £5-7958333 ; £5-7958. | 6. £0-6739583 ; £0-6740. | |
| 7. £0-9604167 ; £0-9604. | 8. £2-0802083 ; £2-0802. | |
| 9. £3-5947917 ; £3-5948. | 10. £7-0385417 ; £7-0385. | |
| 11. 0-306 ton. | 12. 0-216 ton. | 13. 0-146 ton. |
| 14. 0-047 ton. | 15. 0-479 ton. | 16. 1-671 tons. |
| 17. 0-897767857 ton, 0-89777 ton. | 18. 0-293303571 ton, 0-29330 ton. | |
| 19. 0-082589286 ton, 0-08259 ton. | 20. 2-016517857 ton, 2-01652 ton. | |
| 21. 8-599107143 ton, 8-59911 ton. | 22. 0-031808036 ton, 0-03181 ton. | |
| 23. 0-17022315 ton, 0-17022 ton. | 24. 3-983839286 ton, 3-98384 ton. | |
| 25. 7-677991071 ton, 7-67799 ton. | 26. 0-880970982 ton, 0-88097 ton. | |
| 27. 0-780113636 mile, 0-78011 mile. | | |
| 28. 0-419886364 mile, 0-41989 mile. | | |
| 29. 0-464545455 mile, 0-46455 mile. | | |
| 30. 2-202954545 miles, 2-20295 miles. | | |
| 31. 0-433750000 mile, 0-43375 mile. | | |
| 32. 0-844125000 mile, 0-84413 mile. | | |
| 33. 1-045113636 mile, 1-04511 mile. | | |
| 34. 4-235375000 mile, 4-23538 mile. | | |
| 35. (i) £27 18s. 1½d., (ii) £112 4s. 4½d., (iii) £4898 8s. 9d. | | |
| 36. 1 £0-059375 | 37. 1 £0-14375 | |
| 2 £0-118750 | 2 £0-28750 | |
| 3 £0-178125 | 3 £0-43125 | |
| 4 £0-237500 | 4 £0-57500 | |
| 5 £0-296875 | 5 £0-71875 | |
| 6 £0-356250 | 6 £0-86250 | |
| 7 £0-415625 | 7 £1-00625 | |
| 8 £0-475000 | 8 £1-15000 | |
| 9 £0-534375 | 9 £1-29375 | |

Cost (i) £11 8s. ; (ii) £76 19s. ; (iii) £348 15s. 4½d. Cost (i) £66 11s. 1½d. ; (ii) £269 7s. 9d. ; (iii) £818 18s. 10½d.

38. 1	£0.23125
2	£0.46250
3	£0.69375
4	£0.92500
5	£1.15625
6	£1.38750
7	£1.61875
8	£1.85000
9	£2.08125

Cost = £216 9s.

39. 1	£0.0115476
2	£0.0230952
3	£0.0346429
4	£0.0461905
5	£0.0577381
6	£0.0692856
7	£0.0808333
8	£0.0923810
9	£0.1039286

Cost = £88 16s. 3d.

40. £321,467 5s. 4d. 41. £135,148 6s. 42. £416 11s. 3d.
 43. £342 6s. 7d. 44. £256 8s. 11½d.
 45. 2s. 4¾d. = £0.11979167; Cost = £13,416 13s. 4d.
 46. 0.6745833 oz. Troy; Value = £41 9s.
 47. £123 14s. 5d. 48. 468 tons 3 cwt. 19.5 lb.
 49. £162 3s. 3d. 50. 42 tons 11 cwt. 2 qr. 25 lb.
 51. 6s. 3d. 52. (i) 12.5 ft. too short, (ii) 29.2 ft. too long.

EXERCISES 5. (Page 84)

1. 9 : 11. 2. 3 : 7. 3. 15 : 16. 4. 3 : 5.
 5. 11 : 15. 6. 4 : 7. 7. 15 : 19. 8. 3 : 4.
 9. 12 : 5. 10. 1 cwt. 3 qr. 19 lb. 3 cwt. 8 lb.
 11. £1 5s. 2½d. 12. £18. 13. 13s. 5d. more.
 14. 15s. 8d.; £1 1s. 6½d.; £1 13s. 3½d. 15. (i) 5 : 12; (ii) 36 lb.
 16. £11 2s. 9d. 17. 3 : 4. 18. 3 : 5.
 19. (i) £20,111; (ii) 17 : 400.
 20. (i) £1119 5s.; (ii) A, £213 5s. 3d.; B, £325 12s.; C, £580 7s. 9d.
 21. 4.62%. 22. 13.2%. 23. 84.73%.
 24. 37.5%; 42.5%; 6.72%. 25. 8.23%.
 26. (i) 3.6%; (ii) 2.7%; (iii) 4.8%. 27. 51.8%. 28. 3.58%.
 29. 18.44%. 30. 12% increase. 31. £2 8s.
 32. £3 3s. 4d. 33. 3½d. per lb. 34. (i) £4 1s. 3d.; (ii) £4 10s.
 35. £400. 36. 236 units. 37. 3½d. per lb. 38. 6½%.
 39. £852 8s. 6d. 40. £85. 41. £537 9s. 4½d.
 42. £16 19s. 9d.; 3d. less. 43. 3%. 44. (i) 14.7%; (ii) 15.2%.
 45. £260 3s. 1d.; 2.60%. 46. £40. 47. 40%.
 48. A, £704 19s. 9d.; B, £862 0s. 3d. 49. £117.
 50. 26⅔%. 51. 20%. 52. 10%. 53. 72%; 18s. 9d.
 54. 35.3%. 55. 6¼%. 56. (i) 25 gallons, (ii) 4s. 2d.
 57. 10%. 58. 33⅓%. 59. 20.2%.
 60. (i) 26⅔%; (ii) 21⅓%. 61. 35%. 62. 2s. 5d. per lb.

63. 1s. $1\frac{1}{2}$ d. per lb. 64. (i) 160,834 ; (ii) 245,830.
 65. £1,608,208,318 ; £267,513,058. 66. £1 8s. $0\frac{1}{2}$ d.
 67. 2s. $8\frac{1}{4}$ d. 68. 3s. $2\frac{1}{2}$ d. 69. 2s. per lb.
 70. 24%. 71. (i) 91·30%, (ii) 41·18%.
 72. (i) $59\cdot2\%$, (ii) $76\cdot2\%$. 73. $19\cdot2\%$. 74. 3 months.
 75. (i) £97 4s. 2d. ; (ii) £3 9s. 11d. 76. 14 years. 77. £50,398·95.
 78. 36 days ; £1 11s. 5d. per day. 79. (i) $12\frac{1}{2}\%$; (ii) £91,852.
 80. (i) £4385, £4152, £4438 ; (ii) 4% below.

EXERCISES 6. (Page 104)

1. £2 2s. 9d. 2. £1626 3s. 9d. 3. 7s.
 4. 8s. 9d. 5. £5 7s. 1d. 6. £158 14s. 1d.
 7. £2 10s. 11d. 8. £55 13s. 9d. 9. £213 17s. 6d.
 10. £1822 14s. 9d. 11. (i) £104 12s. 9d. ; (ii) £350 19s. 3d.
 12. £48 3s. 7d. 13. £290. 14. $5\frac{3}{4}\%$ per annum.
 15. $4\frac{1}{2}\%$ per annum. 16. £520. 17. Not later than February 12th.
 18. £32 12s. 19. £460. 20. $4\frac{3}{4}\%$ per annum.
 21. £186 12s. 7d. 22. £594 7s. 1d. 23. $4\frac{1}{2}\%$ per annum.
 24. $4\frac{3}{4}\%$ per annum. 25. 8s. 5d. 26. £65.
 27. $4\frac{1}{2}\%$ per annum. 28. £2 9s. 6d. more. 29. £6 4s. 10d.
 30. March 5th. 31. £46 5s. 11d. 32. £459 at $3\frac{3}{4}\%$ and £270 at $4\frac{1}{4}\%$.
 33. £206 3s. 6d.

EXERCISES 7A. (Page 111)

1. £24 12s. 9d. 2. £10 10s. 3. £2 2s. 6d.
 4. £2 0s. 3d. 5. £6 5s. 9d. 6. £14 11s. 7d.
 7. £88 13s. 8d. 8. £5 11s. 9. £3 6s.
 10. 2194·09 kroner. 11. $5\frac{3}{4}\%$ per annum. 12. £575 12s. 1d.
 13. May 6th. 14. August 12th. 15. $5\frac{1}{2}\%$ per annum.
 16. £411 19s. 5d. 17. June 20th. 18. £1 18s. 2d.
 19. $3\frac{3}{4}\%$ per annum. 20. £958 6s. 8d. 21. £3643 15s.
 22. £239 3s. 4d. 23. £6 8s. 24. 4s. 6d.
 25. £269 13s. 9d. 26. £91 10s.

EXERCISES 7B. (Page 117)

1. £15 12s. 2. £3 2s. 3d. 3. £5 15s. 6d. 4. £2 18s. 11d.
 5. £85 12s. 10d. 6. (i) £24 10s. ; (ii) £3577. 7. $4\frac{1}{4}\%$ per annum.
 8. May 24th. 9. $3\frac{3}{4}\%$ per annum.
 10. (i) £607 14s. 6d. ; (ii) £599 8s. ; (iii) £599 10s. 3d.
 11. 1s. 7d. 12. (i) £2801 7s. 6d. ; (ii) £25 17s. 1d. ; (iii) £38 7s. 6d.
 13. May 5th ; £12 9s. 14. 8% per annum.

EXERCISES 7c. (Page 124)

- | | | |
|-------------------------------------|---|--------------------|
| 1. November 21st. | 2. October 25th. | 3. September 26th. |
| 4. June 4th. | 5. June 20th. | 6. July 8th. |
| 7. May 4th. | 8. April 1st. | 9. July 5th. |
| 10. (i) July 7th ; (ii) £5 15s. 6d. | 11. June 8th. | 12. May 27th. |
| 13. May 29th. | 14. (i) June 13th ; (ii) 6404.58 dollars. | |

EXERCISES 8. (Page 137)

- | | | |
|--|------------------------------|--------------------|
| 1. £84 5s. 8d. | 2. £711 15s. 2d. | 3. £307 12s. 3d. |
| 4. £2211 9s. 5d. | 5. £477 4s. 9d. | 6. £3 6s. 3d. |
| 7. £1783 7s. 11d. | 8. £173 4s. 3d. | 9. £699 15s. 4d. |
| 10. £993 4s. 5d. | 11. £1078 10s. 11d. | 12. £41 4s. 9d. |
| 13. (a) gives the greater amount by 2s. 8d. | | |
| 14. £183 17s. 5d. | 15. £515 13s. 3d. | 16. £40 11s. 8d. |
| 17. £33 3s. | 18. £23 17s. 2d. | 19. £1311 5s. 6d. |
| 20. (i) $5\frac{3}{16}\%$ per annum ; (ii) £8 13s. 4d. | | 21. £318 8s. 2d. |
| 22. £84 9s. 3d. | 23. £2626 11s. 3d. | 24. £2 1s. 8d. |
| 25. £2835 7s. 5d. | 26. £70 11s. 8d. | 27. £516 15s. 11d. |
| 28. £2382 8s. 11d. | 29. £287 3s. 5d. | 30. £1600, £1681. |
| 31. £753 19s. 8d. ; £660 14s. 3d. | 32. £291 15s. 5d. | |
| 33. He lost £24 7s. | 34. £380 9s. 3d. | 35. £505. |
| 36. After 7 years ; £490 3s. 10d. | 37. £5966 1s. 7d. | |
| 38. After 6 years ; £371 11s. | 39. (i) 25% ; (ii) £2048. | |
| 40. (i) 20% ; (ii) £3255 ; (iii) £1066 12s. | | |

EXERCISES 9. (Page 154)

- | | | |
|---|---|---------------------------------------|
| 1. £9 6s. 10d. | 2. He gains £5 14s. 7d. | 3. $12\frac{1}{2}\%$. |
| 4. $3\frac{1}{2}\%$ per annum. | 5. 3.2% . | 6. £3 13s. 5d. |
| 7. (i) $A=3\frac{1}{2}\%$, $B=2\frac{3}{4}\%$, $C=4\frac{1}{4}\%$. (ii) Total yield = 3.76% . | | |
| 8. (i) £4 7s. 5d., (ii) £3 6s. 8d. ; 61. | 9. £15. | 10. 16s. 8d. |
| 11. £4512 17s. 2d. | 12. £2 10s. | 13. £14,400 stock ; $83\frac{1}{3}$. |
| 14. 114 ; £120. | 15. $97\frac{1}{2}$. | 16. £139 3s. 4d. |
| 17. 4.75% ; Income increased by £21. | | |
| 18. (i) £3032 2s. 10d. ; (ii) £2380 ; (iii) £2260. (iii) gives the greatest yield. | | |
| 19. Income increased by £16. | 20. Income decreased by £33 13s. 3d. | |
| 21. £41 6s. 6d. | 22. £2697 in $3\frac{1}{4}\%$, £3627 in $5\frac{1}{2}\%$. | |
| 23. £416 in $6\frac{1}{4}\%$, £520 in $5\frac{1}{4}\%$. | 24. £728 in $4\frac{3}{4}\%$, £672 in $5\frac{1}{2}\%$. | |
| 25. £2925 in $3\frac{1}{2}\%$, £3450 in $5\frac{1}{2}\%$. | 26. £1479 in $3\frac{1}{2}\%$, £1508 in $4\frac{1}{2}\%$. | |
| 27. £6300. | 28. £2880 in 4% , £2580 in 5% ; £183. | |

29. (i) £1912 in $3\frac{1}{2}\%$, £1673 in $4\frac{1}{4}\%$; (ii) 4.06% .
 30. £3300 in $5\frac{1}{2}\%$, £2650 in $6\frac{1}{2}\%$. 31. £3500 in $7\frac{1}{2}\%$, £1166 $\frac{2}{3}$ in $3\frac{1}{2}\%$.
 32. £4000 stock.
 33. A takes all the $3\frac{3}{4}\%$ stock + £625 of the $3\frac{1}{2}\%$; B takes £6625 of the $3\frac{1}{2}\%$ stock. Each income = £231 17s. 6d.
 34. $6\frac{2}{7}\%$. 35. 22.7% . 36. £900 stock. 37. £17,460.
 38. $12\frac{1}{2}\%$, £14 15s. 10d.

EXERCISES 10. (Page 171)

1. (i) 226,576 sq. ft.; (ii) 5.20 acres. 2. £2200 per acre.
 3. 58 yd. 2 ft.; £46 4s. 4. £26 0s. 10d.
 5. £2 17s. 6. 18s. per sq. yd. 7. $4\frac{1}{2}$ d. per sq. ft.
 8. 8 pieces. 9. £23 16s. 10. 13.5 sq. in.
 11. 4.60 in. 12. 34 sq. yd.; 8.79 acres. 13. 17.5 sq. in.
 14. £25 13s. 15. 16 pieces.
 16. The second is the cheaper by nearly 1d. in the £. 17. 24 chains.
 18. 4 in. to 1 mile. 19. 232.5 acres. 20. 45 plots; £357 15s.
 21. 6 acres. 22. 15.3 acres. 23. 25 acres.
 24. 435 revolutions. 25. 13 miles per hour. 26. 53 sq. in.
 27. 8.56 sq. ft. 28. $1\frac{1}{8}$ sq. ft. 29. 2.2 sq. ft.
 30. (i) £241 5s. 4d.; (ii) £147 2s. 6d. 31. 16.1 acres.

EXERCISES 11A. (Page 178)

1. 79. 2. 467. 3. 8793. 4. 49.7.
 5. 61.09. 6. 0.093. 7. $27\frac{2}{3}$. 8. $94\frac{1}{2}$.
 9. $25\frac{4}{7}$. 10. 21 in. 11. 457 m. = 500 yd.
 12. £2 11s. 13. 4 in. to 1 mile. 14. $d = 110.5$ in.
 15. $2\frac{3}{4}\%$ per annum. 16. $x = \frac{9}{23}$. 17. $r = 16$.
 18. 15% . 19. $3\frac{3}{4}\%$ per annum.
 20. (i) $2\frac{1}{2}\%$ per annum, (ii) £4800.

EXERCISES 11B. (Page 185)

1. 5.38516. 2. 73.258. 3. 237.655. 4. 0.85.
 5. 0.942. 6. 7.6. 7. 3.22. 8. 0.957.
 9. 0.38. 10. 0.176. 11. 1 inch to 4 miles.
 12. 25 inches to 1 mile.
 13. 0.000163; $1\frac{6}{37} = 1.162162\dots$, $\sqrt{1.351} = 1.162325\dots$
 14. 3.4 per thousand. 15. $r = 19.7$.
 16. To four places; $\sqrt{9.87} = 3.14165\dots$ 17. Radius = 9.30 feet.
 18. (i) 385 yards; (ii) £121 15s.

19. To five places. $3\frac{370}{1197} = 3\cdot3166248\dots$, $\sqrt{11} = 3\cdot3166247\dots$,
 $3\frac{120}{370} = 3\cdot3166226\dots$.
20. $4\cdot4\%$. 21. To three places. $\sqrt{1\cdot53} = 1\cdot23693\dots$, $\frac{4}{38} = 1\cdot23684\dots$.
22. 390 yards. 23. (i) $BC = 682\cdot44$ yd., (ii) Area = $42\cdot65$ acres.
24. $40\cdot7$ square feet. 25. (i) $AF = 135$ yards, (ii) $22\cdot5$ acres.
26. $2\frac{1}{4}\%$ per annum. 27. (i) $3\frac{1}{4}\%$ per annum, (ii) £825.
28. $3757\cdot2$ yards.

EXERCISES 12. (Page 197)

- | | | |
|--|---------------------------|-------------------------------|
| 1. $492\frac{3}{16}$ gallons. | 2. 4356 bricks. | 3. 7·2 gm. per c.c. |
| 4. 3·2 ft. | 5. 3·2 in. | 6. $64\cdot4$ kgm. |
| 7. 159 lb. 14 oz. | 8. 4 ft. | 9. 10·2 lb. |
| 10. (i) 4·5 ft., (ii) 291 sq. ft. | | |
| 11. Cuboid, 3 ft. 9 in. by 2 ft. 3 in. by 1 ft. $1\frac{1}{2}$ in. | Cube, 2 ft. 3 in. edge. | |
| 12. $6\frac{7}{8}$ in. | 13. 17·2 cm. | 14. 2 ft. 11·30 in. |
| 15. 19 loads. | 16. 1·32 in. | 17. 7·7 km. per hour. |
| 18. $3\frac{1}{2}$ in. | 19. 199·1 vessels. | 20. $37\cdot84\frac{0}{10}$. |
| 21. 2·83 in. | 22. 240. | 23. 42·4 ft. |
| 24. 263·76 sq. ft. | | |
| 25. 3·1 in. | 26. £6 3s. 9d. | 27. 33·3 in. |
| 28. 52 lb. | 29. 15·4 ft. | 30. 9·9 lb. |
| 31. 21 lb. 1 oz. | 32. 4 lb. 7 oz. | 33. 1·58 kgm. |
| 34. 3·13 ft. | 35. 3 mm. | |
| 36. 2·58 cm. | 37. 498 lb. | 38. 708·3 lb. per cu. ft. |
| 39. 25% . | 40. 5 francs 76 centimes. | 41. $14\frac{35}{44}$ oz. |

EXERCISES 13. (Page 213)

- | | | | |
|---|--|---|---------------------|
| 1. 1. | 2. 1000. | 3. 0·1. | 4. $\frac{4}{81}$. |
| 5. (a) x^5 ; (b) 1; (c) $\frac{1}{27}$; (d) 4. | 6. (a) x^3 ; (b) $\frac{1}{4}$; (c) -3. | | |
| 7. 72·6. | 8. 10·06. | 9. 33·52. | 10. £18 16s. |
| 11. 49·71 chains. | 12. 35·87. | 13. (i) 177·3; (ii) 9·272; (iii) 1·453. | |
| 14. 0·4071. | 15. 0·0864. | 16. 0·6721. | |
| 17. (a) 18·12; (b) 0·3003. | 18. 0·26 cm. | 19. 24·89. | |
| 20. 1·042. | 21. 0·9396. | 22. 0·9202. | |
| 23. (i) 1·343; (ii) 13,460 tons. | 24. 34·33 tons. | | |
| 25. (i) 2·47 acres; (ii) 2·59 sq. km. | 26. 2·175. | | |
| 27. (a) 2·904; (b) 1·309 cu. yd. | 28. 2·653. | 29. $r = 3\cdot5$. | |
| 30. 3 in. | | | |

EXERCISES 14. (Page 225)

- | | | | |
|--|------------------|------------------|--------------|
| 1. £1025 16s. 9d. | 2. £141 14s. 2d. | 3. £1428 2s. 5d. | 4. £429 10s. |
| 5. (a) £1390 16s. 5d.; (b) £1601 13s. 8d.; (c) £200 17s. 3d. | | | |
| 6. 3% per annum. | 7. 7 years. | 8. £7478 6s. | |

9. 0.44% per annum. 10. 18 years; £195 18s.
 11. (a) £696 1s. 8d.; (b) £157 7s. 3d. 12. 14 years.
 13. $2\frac{1}{4}\%$ per annum. 14. £450. 15. 17% per annum.
 16. £1003 8s. 2d. 17. $3\frac{1}{2}\%$ per annum. 18. 9.3% per annum.
 19. $4\frac{1}{2}\%$ per annum. 20. 19 years. 21. In 1954.
 22. $4\frac{1}{2}\%$ per annum. 23. £968 11s. 24. 16 years.
 25. 21 years of age. 26. £512 2s. 1d.

EXERCISES 15. (Page 234)

1. $89\frac{1}{2}$. 2. $12\frac{1}{12}$; $169\frac{7}{12}$. 3. 225. 4. (i) £220; (ii) £340.
 5. (i) -91; (ii) 0. Terms equidistant from the ends of the series are equal but opposite in sign, and therefore cancel each other out.
 6. £3400. 7. (i) £1 15s.; (ii) £1215 10s.; (iii) £93 10s. per week.
 8. $40\frac{1}{2}\%$ per annum. 9. $21\frac{9}{11}\%$ per annum. 10. 16% per annum.
 11. $10\cdot2\%$ per annum. 12. $8\cdot4\%$ per annum. 13. 13.0% per annum.
 14. 48% per annum. 15. 25% per annum. 16. $14\frac{2}{7}\%$ per annum.
 17. $4\cdot7\%$ per annum. 18. 6.5% per annum. 19. $8\frac{1}{3}\%$ per annum.
 20. £28 14s. 21. 6s. per month.

EXERCISES 16. (Page 246)

1. $2\frac{233}{4096}=2\cdot057$. 2. 761,543. 3. 1724.123.
 4. (i) $5\frac{1}{2}\%$ per annum; (ii) £7619. 5. $2\frac{2}{3}=2\cdot92$; 18th term.
 6. £3439. 7. $2\frac{2}{5}=2\cdot4$. 8. £2562 9s. 6d.
 9. £2697 11s. 2d. 10. £1447. 11. £478.
 12. £4 14s. 13. £367 nearly. 14. £14 1s.
 15. £27 11s. 4d. 16. £4305. 17. £1578.89 = £1579 to the nearest £.
 18. £54 19s. or £55. 19. £753 19s. 8d.; £660 14s. 3d.
 20. £4925 8s. 9d. 21. 16 years. 22. 17.5 years.
 23. 19 years. 24. £295. 25. £330 7s. 4d.
 26. 17 years.

EXERCISES 17. (Page 262)

1. £99 11s. 7d. 2. £15 11s. 3. £260 8s. 2d.
 4. £166 8s. 5. £613 6s. 6. £2042 7s. 9d.
 7. £523 13s. 2d. 8. £302. 9. $4\frac{1}{4}\%$ per annum.
 10. £398 10s. 7d. 11. £25 1s. 10d. 12. £80 11s.
 13. £1129 12s. 14. £2252. 15. £14 18s. 6d.
 16. £379 10s. 17. £100 8s. 7d. 18. £3061 14s. 1d.
 19. 60 payments. 20. 14 years. 21. 18 payments.
 22. 47 payments. 23. £723 19s. 2d. 24. £7 16s. 9d.

EXERCISES 18. (Page 283)

1. 24.4% nearly. 16.91% .
2. (i) $1\frac{1}{8}$ sq. ft. = 168 sq. in. (ii) $3\frac{1}{8}$ sq. ft. = 448 sq. in.
3. 66 sq. in. 4. 2156 c.c. 5. 5.9 cu. ft. 6. 980.
7. (i) 5 tons 10 cwt. 1 qr. 27 lb. (ii) £10 6s. 3d.
8. (i) 7 cm.; (ii) Surface of sphere = 423.5 sq. cm. Surface of cylinder = 346.5 sq. cm. Difference = 77 sq. cm.
9. $11\frac{1}{4}$ gallons. 10. $12\frac{3}{8}$ cu. ft.
11. (i) 809.58 cu. in.; (ii) 110.88 sq. in.
12. 2.5 gm. per c.c. 13. 64.35 lb.
14. (i) 902 cu. in.; (ii) 31.1 lb. 15. Diameter = 0.8 in.
16. 84 balls. 17. 8.85 in.
18. (i) 8.22 ft.; (ii) 105 sq. ft.; (iii) 126 lb.
19. 3.6 in. 20. 4.8 in. 21. 0.36 in. 22. 160.1 lb.
23. 4.15 in. 24. 9.8 in. 25. 38 lb.
26. 13.82 cm.; 72.35% . 27. 18.2 gallons per minute.
28. 18.5 pints. 29. 32 cu. in. 30. 422.7 litres. 31. 3.6 gallons.
32. 37.75 gallons.

EXERCISES 19. (Page 298)

1. (i) 1s. 5d.; (ii) 3s. 5d.; (iii) 3s. 7d. 2. 44.2 francs per kgm.
3. (i) £3 8s. 1d.; (ii) 4 tons 6 cwt. 4. $\frac{2}{3}$. 5. (i) £1 3s. 9d.; (ii) 66.
6. (i) £55; (ii) 28 horse-power. 7. 1s. 7d.; 2s. 5d.
8. (i) 1935.36; (ii) £62.7 thousand.
9. (i) March-April; (ii) April; (iii) May-June, August-September.
10. (i) 1931.32; (ii) £787 millions. 11. £46. 12. $T = 14$.
13. (i) £66.8; (ii) £57.9. 14. (i) £2.51; (ii) 13 years.
15. 1.3; 30% per annum. 16. (i) 16 years; (ii) 14 yr. and $19\frac{1}{2}$ yr.
17. (i) 94.2; (ii) £2 19s. 1d. 18. (i) £3 18s.; (ii) 28 years.
19. (i) 20.2 years; (ii) At the age of 57.
20. (i) 23 years; (ii) £1950. 21. (i) £33 4s.; (ii) 14 years.

EXERCISES IN SECTION 20.1. (Page 305)

- (i) £12 11s. $0\frac{1}{2}$ d.; (ii) £24 17s. $11\frac{1}{2}$ d.; (iii) 9s. $8\frac{1}{2}$ d.; (iv) £192 0s. $4\frac{1}{2}$ d.;
(v) £11 7s. 6d.

TYPICAL EXAMINATION PAPERS

SECTION A. I. (Page 317)

1. 4 tons 15 cwt. 3 qr. 6 lb. 2. A, £57 10s.; B, £46; C, £34 10s.
3. 5s. 8d. per pair. 4. $3\frac{3}{4}\%$ per annum. 5. £1 7s. 5d.

6. £497 4s. 7. 14 days ; 13s. 9d. per day.
 8. (i) 3 places ; $\sqrt{0.629} = 0.793095\dots$, $\frac{2.3}{2.9} = 0.793103\dots$ (ii) 0.130.
 9. £1 2s. 6d. per week. 10. 283.5 cubic inches.

II. (Page 318)

- Part I. 1. (i) £9552 1s. 2d. ; (ii) £11,826 10s. 2d. ; (iii) £13,029 3s. 6d.
 2. (i) £4 16s. 9d. ; (ii) £7 3s. 8d. 3. £0.88125. 4. $\frac{1}{2}$.
 5. 4s. 4 $\frac{1}{2}$ d. 6. £2 7s. 11 $\frac{3}{4}$ d. 7. 16. 8. $\frac{2}{3}$.
 9. 9s. 1d. 10. 20%.
- Part II. 1. £1036 17s. 6d. 2. 1 cwt. 2 qr. 17 lb.
 3. 8.8 inches. 4. £659 12s. 8d. 5. (i) 25% ; (ii) 33%.
 6. £19 10s. 7. 1 sq. yd. = 0.836 sq. m. 8. £1380 3s. 2d.

III. (Page 320)

- | | £ | s. | d. | | £ | s. | d. |
|----------------|-------|----|----|------|------|----|----|
| Part I. 1. (i) | 847 | 8 | 8 | (ii) | 1365 | 11 | 6 |
| | 304 | 18 | 2 | | 1414 | 19 | 5 |
| | 943 | 4 | 1 | | 1612 | 15 | 2 |
| | 504 | 13 | 8 | | | | |
| | 997 | 0 | 0 | | | | |
| | 263 | 16 | 8 | | | | |
| | 532 | 4 | 10 | | | | |
| (iii) | £4393 | 6 | 1 | | | | |
2. (i) £12 11s. 9d. ; (ii) 0.48125 ; (iii) 7s. 5 $\frac{1}{2}$ d. ; (iv) 55,638 ; (v) 0.584 ; (vi) 5%.
3. (i) £2 9s. 4d. ; (ii) 5s. ; (iii) £2 13s. per ton ; (iv) 1s. 1 $\frac{3}{4}$ d. ; (v) 3s. 6d. ; (vi) 13s. 1 $\frac{1}{2}$ d.
- Part II. 1. (a) £5 ; (b) 0.144. 2. £141 19s. 6d.
 3. £1 3s. 10d. per ton. 4. 6s. 11d. 5. 2.47 acres.
 6. (i) 3 cwt. 2 qr. 18 lb. ; (ii) £4 5s. 5d. 7. 10s. 2d. in the £.
 8. 2 $\frac{3}{4}$ % per annum.

SECTION B. I. (Page 322)

1. £472 3s. 2d. ; 20.2%. 2. 23%.
 3. (i) 9% ; (ii) 32.1%. 4. £580 14s. 5. £2887.
 6. (i) 93 $\frac{3}{4}$; (ii) £5500. 7. A, £1281 ; B, £1007 ; C, £333.
 8. £98 1s.

II. (Page 323)

1. (i) 4 places ; $\frac{1.17}{2.53} = 0.462450\dots$, $\sqrt{0.21386} = 0.462449\dots$; (ii) £3 8s. 7d., £250 6s. 7d.
 2. £71,891,528 ; £402,850,842 ; 11.59% increase ; 17.77% increase.
 3. 5s. per pair. 4. £2133 6s. 8d. 5. 6d. in the £.

6. £4147 in $4\frac{1}{4}\%$, £6409 in $3\frac{1}{2}\%$.
 7. (i) From 1930-31 to 1931-32; £33·1 millions; (ii) £270·5 millions.
 8. 88·825 sq. in. 9. £2185. 10. 175 francs to the £.

III. (Page 325)

1. £729 5s. 8d. 2. £620,071 11s. 5d. 3. 2·6 per cent.
 4. £393 13s. 6d. 5. 1·233. 6. £28 17s. 6d.; £1 3s. 4d.
 7. 6080 shares; £9626 13s. 4d. 8. 11s. 0 $\frac{1}{2}$ d. 9. £1327 1s. 8d.
 10. 58·8 in.

IV. (Page 326)

1. (i) 0·192; (ii) 4s. 3d. per yard. 2. $1\frac{1}{4}\%$ increase. 3. 2·93 in.
 4. £7 12s. 3d. 5. (i) £917 7s. 2d.; (ii) £37 8s.
 6. A, £696 12s.; B, £453 5s. 7. £1591 in $3\frac{3}{4}\%$; £6149 in $5\frac{1}{4}\%$.
 8. The graph is a straight line. (i) £2 5s.; (ii) 17s. 9d.

PAPERS AT AN ADVANCED STAGE

I. (Page 328)

1. 178 francs = £1. 2. $1\frac{1}{2}$ d. in the £. 3. £453 18s. 7d.
 4. July 2nd. 5. $12\frac{1}{2}\%$ per annum. 6. £6711 14s.
 7. 26. 8. 31% profit. 9. £2821.
 10. $2\frac{1}{4}\%$ per annum.

II. (Page 329)

1. £5 11s. 3d. 2. $8\frac{5}{8}\%$. 3. May 24th. 4. £3515 4s.
 5. Diameter = 19·65 in. Surface area = 8·43 sq. ft.
 6. 12s. 7d. in the £. 7. 7 lb.; 42·35 $\frac{0}{10}$. 8. 12s. 11d. in the £.
 9. £99 3s. 3d.; 13s. 3d. 10. £64 4s. 8d.
 11. (i) $12\frac{1}{2}\%$ on cost; $11\frac{1}{9}\%$ on sales. 12. 2·92 cu. ft.

III. (Page 331)

1. (i) $12\cdot3\%$, $7\cdot9\%$; (ii) $44\cdot6\%$, $42\cdot9\%$. 2. 23·775 belgas = £1.
 3. £3 2s. 2d. 4. $12\frac{1}{2}\%$. 5. (i) $\frac{3}{4}$; (ii) £1635.
 6. £580 3s. 6d. 7. £155 13s. 8d.
 8. (i) £3 12s.; (ii) At the age of 29.

